THE COBORDISM GROUP OF HOMOLOGY CYLINDERS

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Abstract. Garoufalidis and Levine introduced the homology cobordism group of homology cylinders over a surface. This group can be regarded as an enlargement of the mapping class group. Using torsion invariants, we show that the abelianization of this group is infinitely generated provided that the first Betti number of the surface is positive. In particular, this shows that the group is not perfect. This answers questions of Garoufalidis-Levine and Goda-Sakasai. Furthermore we show that the abelianization of the group has infinite rank for the case that the surface has more than one boundary component. These results hold for the homology cylinder analogue of the Torelli group as well.

1. Introduction

Given $g \geq 0$ and $n \geq 0$, let $\Sigma_{g,n}$ be a fixed oriented, connected and compact surface of genus g with n boundary components. We denote by $\mathrm{Hom}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$ the group of orientation preserving diffeomorphisms of $\Sigma_{g,n}$ which restrict to the identity on the boundary. The *mapping class group* $\mathcal{M}_{g,n}$ is defined to be the set of isotopy classes of elements in $\mathrm{Hom}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$, where the isotopies are understood to restrict to the identity on the boundary as well. We refer to [FM09, Section 2.1] for details. It is well known that the mapping class group is perfect provided that $g \geq 3$ [Po78] (e.g., see [FM09, Theorem 5.1]) and that mapping class groups are finitely presented [BH71, Mc75] (e.g., see [FM09, Section 5.2]).

In this paper we intend to study an enlargement of the mapping class group, namely the group of homology cobordism classes of homology cylinders. A *homology cylinder* over $\Sigma_{g,n}$ is roughly speaking a cobordism between surfaces equipped with a diffeomorphism to $\Sigma_{g,n}$ such that the cobordism is homologically a product. Juxtaposing homology cylinders gives rise to a monoid structure. The notion of homology cylinder was first introduced by Goussarov [Go99] and Habiro [Ha00] (where it was referred to as a 'homology cobordism').

By considering smooth (respectively topological) homology cobordism classes of homology cylinders we obtain a group $\mathcal{H}^{\mathrm{smooth}}_{g,n}$ (respectively $\mathcal{H}^{\mathrm{top}}_{g,n}$). These groups were introduced by Garoufalidis and Levine [GL05], [Le01]. We refer to Section 2 for the precise definitions of homology cylinder and homology cobordism. Henceforth, when a statement holds in both smooth and topological cases, we will drop the decoration in the notation and simply write $\mathcal{H}_{g,n}$ instead of $\mathcal{H}^{\mathrm{top}}_{g,n}$ and $\mathcal{H}^{\mathrm{smooth}}_{g,n}$.

It follows immediately from the definition that there exists a canonical epimorphism $\mathcal{H}^{\mathrm{smooth}}_{g,n} \to \mathcal{H}^{\mathrm{top}}_{g,n}$. A consequence of work of Fintushel-Stern [FS90], Furuta [Fu90] and Freedman [Fr82] on smooth homology cobordism of homology 3–spheres

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is that this map is not an isomorphism. In fact, using their results we can see the following:

Theorem 1.1. *Let $g, n \geq 0$. Then the kernel of the epimorphism $\mathcal{H}^{\rm smooth}_{g,n} \to \mathcal{H}^{\rm top}_{g,n}$ contains an abelian group of infinite rank. If $g = 0$, then there exists in fact a homomorphism $\mathcal{F}\colon \mathcal{H}^{\rm smooth}_{0,n} \to A$ onto an abelian group of infinite rank such that the restriction of $\mathcal{F}$ to the kernel of the projection map $\mathcal{H}^{\rm smooth}_{0,n} \to \mathcal{H}^{\rm top}_{0,n}$ is also surjective.*

An argument of Garoufalidis and Levine shows that the canonical map $\mathcal{M}_{g,n} \to \mathcal{H}_{g,n}$ is injective. (See also Proposition 2.4.) It is a natural question which properties of mapping class groups are carried over to $\mathcal{H}_{g,n}$. In particular in [GS09] Goda and Sakasai ask whether $\mathcal{H}^{\rm smooth}_{g,1}$ is a perfect group and Garoufalidis and Levine [GL05, Section 5, Question 9] ask whether $\mathcal{H}^{\rm smooth}_{g,1}$ is infinitely generated (see also [Mo06, Problem 11.4]).

The following theorem answers both questions:

Theorem 1.2. *If $b_1(\Sigma_{g,n}) > 0$, then there exists an epimorphism*

$$\mathcal{H}_{g,n} \longrightarrow (\mathbb{Z}/2)^\infty$$

which splits (i.e., there is a right inverse). In particular, the abelianization of $\mathcal{H}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

Note that Theorem 1.2 also implies that $\mathcal{H}_{g,n}$ is not finitely related, since for a finitely related group its abelianization is also finitely related. Also refer to Remark 5.5 for a slightly more refined statement.

In many cases we can actually strengthen the result:

Theorem 1.3. *If $n > 1$, then there exists an epimorphism*

$$\mathcal{H}_{g,n} \longrightarrow \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

We remark that for the special case of $(g, n) = (0, 2)$, this is a consequence of Levine's work on knot concordance [Le69a, Le69b] since one can easily see that $\mathcal{H}^{\rm top}_{0,2}$ maps onto Levine's algebraic knot concordance group. The general cases of Theorem 1.3 for $n > 2$ and for $n = 2$ and $g > 0$ are new.

In order to prove Theorems 1.2 and 1.3 we will employ the torsion invariant of a homology cylinder, first introduced by Sakasai (e.g., see [Sa06, Section 11.1.2], [Sa08, Definition 6.5] and [GS08, Definition 4.4]). In Section 3 we recall the definition of the torsion of a homology cylinder and we study the behavior of torsion under stacking and homology cobordism. The result can be summarized as a group homomorphism

$$\tau\colon \mathcal{H}_{g,n} \longrightarrow \frac{Q(H)^\times}{A(H)N(H)}.$$

Here $H = H_1(\Sigma_{g,n})$ and $Q(H)^\times$ is the multiplicative group of nonzero elements in the quotient field of the group ring $\mathbb{Z}[H]$. Loosely speaking, $A(H)$ reflects the action of surface automorphisms on H, and $N(H)$ is the subgroup of "norms" in $Q(H)^\times$. (For details, see Section 3.)

An interesting point is that torsion invariants of homology cylinders may be *asymmetric*, in contrast to the symmetry of the Alexander polynomial of knots.

Indeed, in Section 5, we extract infinitely many $(\mathbb{Z}/2)$–valued and $\mathbb{Z}$–valued homomorphisms of $Q(H)^\times/A(H)N(H)$ from symmetric and asymmetric irreducible factors, respectively. Theorems 1.2 and 1.3 now follow from the explicit construction of examples in Section 4.

We remark that Theorem 1.3 covers all the possibilities of the asymmetric case, since it can be seen that for either $n < 2$ or $(g, n) = (0, 2)$ the torsion of a homology cylinder over $\Sigma_{g,n}$ is always symmetric in an appropriate sense (see Section 3.6 for details.)

We remark that our main results hold even modulo the mapping class group—the essential reason is that our torsion invariant is trivial for homology cylinders associated to mapping cylinders. More precisely, denoting by $\langle \mathcal{M}_{g,n} \rangle$ the normal subgroup of $\mathcal{H}_{g,n}$ generated by $\mathcal{M}_{g,n}$, the torsion homomorphism τ actually factors as

$$\tau \colon \mathcal{H}_{g,n} \longrightarrow \frac{\mathcal{H}_{g,n}}{\langle \mathcal{M}_{g,n} \rangle} \longrightarrow \frac{Q(H)^\times}{A(H)N(H)}$$

and Theorems 1.2 and 1.3 hold for $\mathcal{H}_{g,n}/\langle \mathcal{M}_{g,n} \rangle$ as well as $\mathcal{H}_{g,n}$.

In Section 6, we study examples of homology cylinders which naturally arise from Seifert surfaces of pretzel links. We compute the torsion invariant and prove that these homology cylinders span a $\mathbb{Z}^\infty$ summand in the abelianization of $\mathcal{H}_{0,3}$.

Finally in Section 7, we consider the "Torelli subgroup" $\mathcal{IH}_{g,n}$ of $\mathcal{H}_{g,n}$ which is the homology cylinder analogue of the Torelli subgroup of mapping cylinders. We prove that the conclusions of Theorems 1.2 and 1.3 hold for the Torelli subgroup $\mathcal{IH}_{g,n}$, but for a larger group of surfaces. (See Theorem 7.2 for details.) This extends work of Morita's [Mo08, Corollary 5.2] to a larger class of surfaces.

In this paper, manifolds are assumed to be compact, connected, and oriented. All homology groups are with respect to integral coefficients unless it says explicitly otherwise.

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2. Homology cylinders and their cobordism

In this section we recall basic definitions and preliminaries on homology cylinders, the notion of which goes back to Goussarov [Go99], Habiro [Ha00], and Garoufalidis and Levine [GL05], [Le01].

2.1. **Cobordism classes of homology cylinders.** Given $g \geq 0$ and $n \geq 0$ we fix, once and for all, a surface $\Sigma_{g,n}$ of genus g with n boundary components. When g and n are obvious from the context, we often denote $\Sigma_{g,n}$ by Σ. A *homology cylinder* (M, i_+, i_-) *over* Σ is defined to be a 3–manifold M together with injections i_+, $i_-\colon \Sigma \to \partial M$ satisfying the following:

(1) i_+ is orientation preserving and i_- is orientation reversing.
(2) $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$.
(3) $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$.
(4) $i_+, i_-\colon H_*(\Sigma) \to H_*(M)$ are isomorphisms.

Example 2.1.

(1) Let $\varphi \in \mathcal{M}_{g,n}$. Then φ gives rise to a homology cylinder
$$M(\varphi) = (\Sigma_{g,n} \times [0,1]/\sim,\, i_+ = \mathrm{id} \times 0,\, i_- = \varphi \times 1).$$
where $\sim$ is given by $(x, s) \sim (x, t)$ for $x \in \partial\Sigma_{g,n}$ and $s, t \in [0, 1]$. If φ is the identity, then we will refer to the resulting homology cylinder as the *product homology cylinder*.
(2) Let K be a knot of genus g such that the Alexander polynomial Δ_K is monic and of degree $2g$. Let Σ be a minimal genus Seifert surface in the exterior X of K. Then X cut along Σ is a homology cylinder over $\Sigma_{g,1}$ in a natural way (e.g., see [Ni07, Proposition 3.1] or [GS08] for details).

Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma = \Sigma_{g,n}$ are called *isomorphic* if there exists an orientation-preserving diffeomorphism $f\colon M \to N$ satisfying $j_\pm = f \circ i_\pm$. We denote by $\mathcal{C}_{g,n}$ the set of all isomorphism classes of homology cylinders over $\Sigma_{g,n}$. A product operation on $\mathcal{C}_{g,n}$ is given by stacking:
$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$
This turns $\mathcal{C}_{g,n}$ into a monoid. The unit element is given by the product homology cylinder.

Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over Σ are called *smoothly homology cobordant* if there exists a compact oriented *smooth* 4–manifold W such that
$$\partial W = M \cup (-N) \,/\, i_+(x) = j_+(x),\, i_-(x) = j_-(x) \quad (x \in \Sigma),$$
and such that the inclusion induced maps $H_*(M) \to H_*(W)$ and $H_*(N) \to H_*(W)$ are isomorphisms. We denote by $\mathcal{H}^{\mathrm{smooth}}_{g,n}$ the set of smooth homology cobordism classes of elements in $\mathcal{C}_{g,n}$. The monoid structure on $\mathcal{C}_{g,n}$ descends to a group structure on $\mathcal{H}^{\mathrm{smooth}}_{g,n}$, where the inverse of a mapping cylinder (M, i_+, i_-) is given by $(-M, i_-, i_+)$. (We refer to [Le01, p. 246] for details).

If there is a topological 4–manifold W satisfying the above conditions, then we say that (M, i_+, i_-) and (N, j_+, j_-) are *topologically homology cobordant*. We denote the resulting group of homology cobordism classes by $\mathcal{H}^{\mathrm{top}}_{g,n}$. Note that there exists a canonical surjection $\mathcal{H}^{\mathrm{smooth}}_{g,n} \to \mathcal{H}^{\mathrm{top}}_{g,n}$. We will see in the following sections that this map is in general not an isomorphism.

Remark 2.2. In the original papers of Garoufalidis and Levine [GL05, Le01] and in [GS09] the authors focus on the smooth case and denote the group $\mathcal{H}^{\mathrm{smooth}}_{g,n}$ by $\mathcal{H}_{g,n}$.

2.2. **Examples.** In this section we will discuss three types of examples:

(1) surface automorphisms,
(2) homology cobordism classes of integral homology spheres, and
(3) concordance classes of (framed) knots in integral homology spheres.

Surface automorphisms and homology cylinders. First recall that $\varphi \in \mathcal{M}_{g,n}$ gives rise to a homology cylinder $M(\varphi)$. Note that if $\varphi, \psi \in \mathcal{M}_{g,n}$, then $M(\varphi) \cdot M(\psi)$ is isomorphic to $M(\varphi \circ \psi)$. In particular the map $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n}$ descends to a morphism of monoids. Proposition 2.4 below says that we can view the mapping class group $\mathcal{M}_{g,n}$ as a subgroup of $\mathcal{H}_{g,n}$. To prove the proposition, we need the following folklore theorem:

Theorem 2.3. *Let Σ be a surface with one boundary component, possibly with punctures. Let $* \in \partial\Sigma$ be a base point. Suppose $h\colon \Sigma \to \Sigma$ is a homeomorphism with the following properties:*

(1) *h is the identity on $\partial\Sigma$,*
(2) *$h_* : \pi_1(\Sigma, *) \to \pi_1(\Sigma, *)$ is the identity, and*
(3) *h fixes each puncture.*

Then h is isotopic to the identity where the isotopy restricts to the identity on $\partial\Sigma$.

Proof. The theorem is well-known, and we only give an outline of the proof. We choose disjoint circles γ_i in Σ based at $*$ such that Σ cut along the γ_i is a punctured disk. Since h_* is the identity, we may assume (after applying an isotopy whose restriction to $\partial\Sigma$ is the identity) that h fixes each γ_i. Now the map on the punctured disk induced from h is isotopic to the identity with $\partial\Sigma$ fixed pointwise. This can be shown, for example, using the fact that the map from the pure braid group to $\pi_1(\Sigma)$ is injective. $\square$

Proposition 2.4. *The map $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n} \to \mathcal{H}_{g,n}$ is injective.*

Proof. The proposition was proved by Garoufalidis and Levine in the case that $n = 1$ and in the smooth category. (See [GL05, Section 2.4] and [Le01, Section 2.1].) Using their arguments partially, we will show that the proposition holds for any $n \geq 0$ (and in the topological category as well).

First note that if $(g, n) = (0, 0)$ or $(0, 1)$, i.e., if $\Sigma_{g,n}$ is a sphere or a disk, then it is well-known that any orientation preserving diffeomorphism is isotopic to the identity. If $(g, n) = (0, 2)$, i.e., if $\Sigma_{g,n}$ is an annulus, then it is known that $\mathcal{M}_{g,n} \cong \mathbb{Z}$ and it injects into $\mathcal{H}_{g,n}$ (for example, this can be shown using the arguments in the subsection below entitled *Concordance of (framed) knots in homology* 3*–spheres*). Therefore we henceforth assume that $(g, n) \neq (0, 0), (0, 1), (0, 2)$.

Suppose $\varphi\colon \Sigma_{g,n} \to \Sigma_{g,n}$ is an orientation-preserving diffeomorphism such that

(1) φ restricts to the identity on $\partial\Sigma_{g,n}$, and
(2) $[\varphi] \in \operatorname{Ker}\{\mathcal{M}_{g,n} \to \mathcal{H}_{g,n}\}$.

Now fix $k \in \{1, 2, \ldots, n\}$ and fix a base point $*$ lying in the k-th component, say ∂_k, of $\partial\Sigma_{g,n}$. We write $\pi := \pi_1(\Sigma_{g,n},\ *)$. We denote by π_l the lower central series of π defined inductively by $\pi_1 := \pi$ and $\pi_l := [\pi, \pi_{l-1}]$, $l > 1$. The argument of Garoufalidis and Levine (which builds on Stallings' theorem [Sta65]) shows that if $M(\varphi)$ is homology cobordant to the product homology cylinder, then $\varphi_* : \pi/\pi_l \to \pi/\pi_l$ is the identity map for any l. Since $\bigcap_l \pi_l$ is trivial, this implies that $\varphi_* : \pi \to \pi$ is the identity.

We denote by $\mathcal{PM}^k_{g,n}$ the set of equivalence classes of orientation-preserving diffeomorphisms $h\colon \Sigma_{g,n} \to \Sigma_{g,n}$ such that

(1) h is the identity on ∂_k, and
(2) h fixes each boundary component setwise,

where we say that two such maps are equivalent if they are related by an isotopy which fixes ∂_k pointwise. Then $[\varphi] = [\text{identity}]$ in $\mathcal{PM}^k_{g,n}$ by Theorem 2.3.

Now consider the map $\mathcal{M}_{g,n} \to \mathcal{PM}^k_{g,n}$. It is known that the n Dehn twists along boundary components of $\Sigma_{g,n}$ generate a central subgroup isomorphic to $\mathbb{Z}^n$ in $\mathcal{M}_{g,n}$, and $\operatorname{Ker}\{\mathcal{M}_{g,n} \to \mathcal{PM}^k_{g,n}\}$ is the subgroup generated by the $n-1$ Dehn twists along all, but the k-th boundary component. (For example, see [FM09, Section 4.2].) Therefore, $[\varphi] \in \bigcap_k \operatorname{Ker}\{\mathcal{M}_{g,n} \to \mathcal{PM}^k_{g,n}\} = \{[\text{identity}]\}$ in $\mathcal{M}_{g,n}$. $\square$

Note that this shows that $\mathcal{H}_{g,n}$ is non-abelian provided that $g > 0$ or $n > 2$. It is straightforward to see that in the remaining cases $\mathcal{H}_{g,n}$ is abelian.

In the following we will see that the cobordism groups of homology cylinders over the surfaces $\Sigma_{0,n}$ with $n = 0, 1, 2$ have been studied under different names for many years.

Homology cobordism of integral homology 3–spheres. We first consider the case $n = 0, 1$. Recall that oriented integral homology 3–spheres form a monoid under the connected sum operation. Two oriented integral homology 3–spheres Y_1 and Y_2 are called smoothly (respectively topologically) cobordant if there exists a smooth (respectively topological) 4–manifold cobounding Y_1 and $-Y_2$. We denote by Θ_3^{smooth} (respectively Θ_3^{top}) the group of smooth (respectively topological) cobordism classes of integral homology 3–spheres.

For $n = 0, 1$ the group $\mathcal{H}^{\text{smooth}}_{0,n}$ (respectively $\mathcal{H}^{\text{top}}_{0,n}$) is naturally isomorphic to the group Θ_3^{smooth} (respectively Θ_3^{top}) (e.g., see [Sa06, p. 59]). Furuta [Fu90] and Fintushel-Stern [FS90] showed that Θ_3^{smooth} has infinite rank. (See also [Sav02, Section 7.2].) On the other hand it follows from the work of Freedman [Fr82] that $\mathcal{H}^{\text{top}}_{0,n} = \Theta_3^{\text{top}}$ is the trivial group. (See also [FQ90, Corollary 9.3C].) This shows in particular that the homomorphism $\mathcal{H}^{\text{smooth}}_{0,n} \to \mathcal{H}^{\text{top}}_{0,n}$ is not an isomorphism for $n = 0, 1$.

Concordance of (framed) knots in homology 3–spheres. We now turn to the case $n = 2$, i.e., homology cylinders over the surface $\Sigma_{0,2}$ which we henceforth identify with the annulus $S^1 \times [0, 1]$. Let $K \subset Y$ be an oriented knot in an integral homology 3–sphere. Let M be the exterior of K. It is not difficult to see there are pairs (i_+, i_-) of maps $\Sigma_{0,2} \to \partial M$ satisfying (1)–(4) in the definition of a homology cylinder and satisfying the condition that $i_+(S^1 \times 0)$ is a meridian of K. Furthermore, the isotopy types of such (i_+, i_-) are in 1–1 correspondence with framings on K. Indeed, the linking number of K and the closed curve $i_-(* \times [0, 1]) \cup i_+(* \times [0, 1])$ gives rise to a canonical 1–1 correspondence between the set of framings and $\mathbb{Z}$. Conversely, a homology cylinder (M, i_+, i_-) over $\Sigma_{0,2}$ determines an oriented knot endowed with a framing in the integral homology sphere, which is given by attaching a 2–handle along $i_+(S^1 \times 0)$ and then attaching a 3–handle.

We say that $K_1 \subset Y_1$ and $K_2 \subset Y_2$ are *smoothly concordant* if there exists a smooth cobordism X of Y_1 and Y_2 such that (X, Y_1, Y_2) is an integral homology $(S^3 \times [0, 1], S^3 \times 0, S^3 \times 1)$ and X contains a smoothly embedded annulus C

cobounding K_1 and K_2. The set of smooth concordance classes of knots in integral homology 3–spheres form a group $\mathcal{C}_{\mathbb{Z}}^{\text{smooth}}$ under connected sum.

We will now see that we can also think similarly of the concordance group of framed knots in integral homology 3–spheres. Note that a concordance (X, C) as above determines a 1–1 correspondence between framings on K_1 and K_2. We say two framed knots in integral homology spheres are *smoothly concordant* if there is a concordance (X, C) via which the given framings correspond to each other. Smooth concordance class of framed knots form a group as well, and it is easily seen that this framed analogue of $\mathcal{C}_{\mathbb{Z}}^{\text{smooth}}$ is isomorphic to $\mathbb{Z} \oplus \mathcal{C}_{\mathbb{Z}}^{\text{smooth}}$. Similarly, if we allow topologically locally flat annuli in topological cobordisms we obtain a group $\mathcal{C}_{\mathbb{Z}}^{\text{top}}$ and its framed analogue $\mathbb{Z} \oplus \mathcal{C}_{\mathbb{Z}}^{\text{top}}$.

As usual we adopt the convention that the group $\mathcal{C}_{\mathbb{Z}}$ can mean either $\mathcal{C}_{\mathbb{Z}}^{\text{smooth}}$ or $\mathcal{C}_{\mathbb{Z}}^{\text{top}}$. It follows easily from the definitions that we have an isomorphism $\mathbb{Z} \oplus \mathcal{C}_{\mathbb{Z}} \to \mathcal{H}_{0,2}$. It follows from the work of Levine [Le69a, Le69b] that $\mathcal{C}_{\mathbb{Z}}$ maps onto the algebraic knot concordance group which is isomorphic to $(\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty \oplus \mathbb{Z}^\infty$, and furthermore the epimorphism from $\mathcal{C}_{\mathbb{Z}}$ to $(\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$ splits. This discussion in particular proves the following special case of Theorem 1.3:

Theorem 2.5. *There exists a split surjection of $\mathcal{H}_{0,2}$ onto $(\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$.*

Note that the subgroup $\mathcal{M}_{0,2}$ in $\mathcal{H}_{0,2} \cong \mathbb{Z} \oplus \mathcal{C}_{\mathbb{Z}}$ is exactly the $\mathbb{Z}$ factor, so that $\mathcal{H}_{0,2}/\mathcal{M}_{0,2} \cong \mathcal{C}_{\mathbb{Z}}$. In particular, Theorem 2.5 holds for $\mathcal{H}_{0,2}/\mathcal{M}_{0,2}$ as well as $\mathcal{H}_{0,2}$.

2.3. **Proof of Theorem 1.1.** Before we turn to the proof of Theorem 1.1 we introduce a gluing operation on homology cylinders. Let M be a homology cylinder over a surface $\Sigma = \Sigma_{g,n}$ and let M' be a homology cylinder over a surface $\Sigma' = \Sigma'_{g',n'}$. Assume that $n, n' > 0$ and fix a boundary component c of Σ and fix a boundary component c' of Σ'. We can now glue Σ and Σ' along c and c' using an orientation reversing homeomorphism. Similarly we can glue M and M' along neighborhoods of $c \subset \partial M$ and $c' \subset \partial M'$. This gives a homology cylinder over $\Sigma \cup_{c=c'} \Sigma'$, which we denote by $M \cup_{c,c'} M'$. We refer to $M \cup_{c,c'} M'$ as the *union of M and M' along c and c'.*

Now let M' be the product homology cylinder over Σ'. The association $M \mapsto M \cup_{c,c'} M'$ gives rise to a monoid homomorphism $\mathcal{C}_{g,n} \to \mathcal{C}_{g+g',n+n'-2}$. We refer to it as an *expansion by Σ' along c.* Note that the expansion map descends to a group homomorphism $\mathcal{H}_{g,n} \to \mathcal{H}_{g+g',n+n'-2}$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We adopt the convention that Θ_3 stands either for Θ_3^{smooth} or Θ_3^{top}. Recall that in Section 2.2 we saw that we can identify $\mathcal{H}_{0,n}, n = 0, 1$, with Θ_3. Also recall that by Section 2.2 the group Θ_3^{smooth} has infinite rank and that Θ_3^{top} is the trivial group.

As we saw above, an expansion by a surface of genus g with $n + 1$ punctures gives rise to a homomorphism $\mathcal{E}\colon \mathcal{H}_{0,1} \to \mathcal{H}_{g,n}$. We also consider the composition $\mathcal{F}\colon \mathcal{H}_{g,n} \to \mathcal{H}_{g,0}$ of n expansions by a disk along all boundary components of $\Sigma_{g,n}$. Loosely speaking, $\mathcal{F}$ is the homomorphism given by filling in the n holes.

Claim. There exists a map $\mathcal{G} : \mathcal{H}_{g,0} \to \mathcal{H}_{0,0}$ such that the composition

$$\Theta_3 = \mathcal{H}_{0,1} \xrightarrow{\mathcal{E}} \mathcal{H}_{g,n} \xrightarrow{\mathcal{F}} \mathcal{H}_{g,0} \xrightarrow{\mathcal{G}} \mathcal{H}_{0,0} = \Theta_3$$

is the identity.

Let $\Lambda = \Sigma_{g,0}$. Let Y be a fixed handlebody of genus g. Given $i = 1, 2$ we write $Y_i = Y$. Since S^3 has a Heegaard decomposition of genus g there exist diffeomorphisms $\varsigma_i : \Lambda \to \partial Y_i$ such that $Y_1 \cup_{\varsigma_1 \circ \varsigma_2^{-1}} Y_2 = S^3$.

We write $H = H_1(\Lambda)$. We will see in Section 3.3 that the action of a homology cylinder M over Λ on $H_1(\Lambda)$ gives rise to a homomorphism $\varphi : \mathcal{M}_{g,0} \to \operatorname{Aut}(H)$. We will write $\operatorname{Aut}^*(H) = \varphi(\mathcal{M}_{g,0})$. We now pick a splitting map $\psi : \operatorname{Aut}^*(H) \to \mathcal{M}_{g,0}$, i.e. a map such that $\varphi \circ \psi$ is the identity on $\operatorname{Aut}^*(H)$. We pick ψ such that $\psi(\mathrm{id}) = \mathrm{id}$.

Note that we can not arrange that ψ is a homomorphism. Now consider the following map

$$\begin{array}{rcl} \mathcal{G} : \mathcal{C}_{g,0} & \mapsto & \{\mathbb{Z}\text{-homology spheres}\} \\ (M, i_+, i_-) & \mapsto & Y_1 \cup_{\varsigma_1 \circ i_+^{-1}} M \cup_{i_- \circ \psi(\varphi(M))^{-1} \circ \varsigma_2^{-1}} Y_2. \end{array}$$

Note that this map is indeed well-defined, i.e. the right hand side is an integral homology 3–sphere. Also note that this map descends to a map $\mathcal{H}_{g,0} \to \Theta_3$. It is easy to verify that

$$\Theta_3 = \mathcal{H}_{0,1} \xrightarrow{\mathcal{E}} \mathcal{H}_{g,n} \xrightarrow{\mathcal{F}} \mathcal{H}_{g,0} \xrightarrow{\mathcal{G}} \mathcal{H}_{0,0} = \Theta_3$$

is indeed the identity map. This concludes the proof of the claim.

Before we continue we point out that the map $\mathcal{G}$ is in general *not* a monoid morphism. We now obtain the following commutative diagram:

$$\begin{tikzcd} \Theta_3^{\text{smooth}} \arrow[r, "="] \arrow[d] & \mathcal{H}_{0,1}^{\text{smooth}} \arrow[r, "\mathcal{E}"] \arrow[d] & \mathcal{H}_{g,n}^{\text{smooth}} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{H}_{g,0}^{\text{smooth}} \arrow[r, "\mathcal{G}"] \arrow[d] & \mathcal{H}_{0,0}^{\text{smooth}} \arrow[r, "="] \arrow[d] & \Theta_3^{\text{smooth}} \arrow[d] \\ \Theta_3^{\text{top}} \arrow[r, "="] & \mathcal{H}_{0,1}^{\text{top}} \arrow[r, "\mathcal{E}"] & \mathcal{H}_{g,n}^{\text{top}} \arrow[r, "\mathcal{F}"] & \mathcal{H}_{g,0}^{\text{top}} \arrow[r, "\mathcal{G}"] & \mathcal{H}_{0,0}^{\text{top}} \arrow[r, "="] & \Theta_3^{\text{top}}. \end{tikzcd}$$

Since $\mathcal{G} \circ \mathcal{F} \circ \mathcal{E} = \mathrm{id}$ we deduce that $\Theta_3^{\text{smooth}} \xrightarrow{\mathcal{E}} \mathcal{H}_{g,n}^{\text{smooth}}$ is injective. Furthermore it follows from $\Theta_3^{\text{top}} = 0$ and from the above diagram that $\mathcal{E}(\Theta_3^{\text{smooth}})$ is contained in the kernel of the projection map $\mathcal{H}_{g,n}^{\text{smooth}} \to \mathcal{H}_{g,n}^{\text{top}}$. Since Θ_3^{smooth} has infinite rank this concludes the proof of the first part of Theorem 1.1.

If $g = 0$, then we can take $\mathcal{G}$ to be the identity map. In particular all maps are homomorphisms, and the homomorphism $\mathcal{F} : \mathcal{H}_{0,n}^{\text{smooth}} \to \Theta_3^{smooth} =: A$ has the desired properties. □

3. Torsion invariants of homology cylinders

3.1. **The torsion invariant.** Let (M, N) be a pair of manifolds. Let $\varphi \colon \pi_1(M) \to H$ be a homomorphism to a free abelian group. We denote the quotient field of $\mathbb{Z}[H]$ by $Q(H)$. Denote by $p \colon \tilde{M} \to M$ the universal covering of M and write $\tilde{N} := p^{-1}(N)$. Then we can consider the chain complex

$$C_*(M, N; Q(H)) = C_*(\tilde{M}, \tilde{N}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H).$$

Let $\mathcal{B}_*$ be a basis for the graded vector space $H_*(M, N; Q(H))$, then we obtain the corresponding torsion

$$\tau_{\mathcal{B}_*}(M, N; Q(H)) \in Q(H)^{\times} := Q(H) - \{0\}.$$

This torsion invariant is well-defined up to multiplication by an element of the form $\pm h$ $(h \in H)$. If (M, N) is $Q(H)$–acyclic, then $\mathcal{B}_*$ is the trivial basis and we just

write $\tau(M, N; Q(H))$. We will not recall the definition of torsion but refer instead to the many excellent expositions, e.g., [Mi66], [Tu01], [Tu02] and [Nic03].

We will several times make use of the following well-known lemma (e.g., see [KLW01, Proposition 2.3] for a proof).

Lemma 3.1. *If $H_*(M, N) = 0$ and if $\pi_1(M) \to H$ is a homomorphism to a free abelian group, then $H_*(M, N; Q(H)) = 0$.*

We also adopt the following notation: given $p, q \in Q(H)$ we write $p \doteq q$ if $p = \epsilon h \cdot q$ for some $\epsilon \in \{-1, 1\}$ and $h \in H$. Put differently, $p \doteq q$ if and only if p and q agree up to multiplication by a unit in $\mathbb{Z}[H]$.

3.2. **Torsion of homology cylinders.** Let (M, i_+, i_-) be a homology cylinder over Σ. Write $H = H_1(\Sigma)$. We normally think of H as a multiplicative group. Denote $\Sigma_\pm = i_\pm(\Sigma) \subset M$. Consider

$$\varphi\colon \pi_1(M) \longrightarrow H_1(M) \xleftarrow{\cong} H_1(\Sigma_+) \xleftarrow{i_+} H = H_1(\Sigma).$$

Since $H_*(M, \Sigma_+) = 0$ it follows from Lemma 3.1 that $H_*(M, \Sigma_+; Q(H)) = 0$. Therefore we can define

$$\tau(M) := \tau(M, \Sigma_+; Q(H)) \in Q(H)^\times.$$

This is referred to as the *torsion* of the homology cylinder $M = (M, i_+, i_-)$.

We will now show that the torsion can be defined in terms of homology. First note that $\mathbb{Z}[H]$ is a unique factorization domain, so that for any finitely generated $\mathbb{Z}[H]$-module M, the *order* of M is defined as an element of $\mathbb{Z}[H]$. We denote it by $\operatorname{ord} M$. (For a precise definition, e.g., see [Tu01, Section 4.2].) Note that $\operatorname{ord} M$ is well-defined up to multiplication by a unit in $\mathbb{Z}[H]$.

Lemma 3.2. *For any homology cylinder M we have $\tau(M) \doteq \operatorname{ord} H_1(M, \Sigma_+; \mathbb{Z}[H])$.*

Proof. It is well known that $\tau(M) = \prod_i (\operatorname{ord} H_i(M, \Sigma_+; \mathbb{Z}[H]))^{(-1)^{i+1}}$ (e.g., see [Tu01, Section 4.2]). Since $H_1(\Sigma_+) \to H_1(M)$ is an isomorphism it follows immediately that $H_0(M, \Sigma_+; \mathbb{Z}[H]) = 0$. One can check that $H_i(M, \Sigma_+; \mathbb{Z}[H]) = 0$ for all $i > 1$, by using Poincaré duality, universal coefficient spectral sequence, and the fact that $H_1(M, \Sigma_-; \mathbb{Z}[H])$ is torsion. $\square$

Remark 3.3.

(1) The torsion of a homology cylinder was first studied by Sakasai [Sa06] and is closely related to the torsion invariants of sutured manifold introduced independently by Benedetti and Petronio [BP01] and [FJR09].

(2) Note that string links give rise to homology cylinders in a natural way. In this context the torsion invariant was first studied by Kirk, Livingston and Wang [KLW01, Definition 6.8].

(3) By Lemma 3.2 (see also [FJR09, Lemma 3.5]) the torsion $\tau(M)$ is in fact an element in $\mathbb{Z}[H]$, furthermore, if $\epsilon\colon \mathbb{Z}[H] \to \mathbb{Z}$ denotes the augmentation defined by $\epsilon(h) = 1$ for $h \in H$, then $\epsilon(\tau(M)) = |H_1(M, \Sigma_+)| = 1$ (e.g., see [Tu86] and [FJR09, Proposition 5]).

(4) By Lemma 3.2, if $M = (M, i_+, i_-)$ is the homology cylinder over an annulus corresponding to a knot $K \subset S^3$ as in Section 2.2, then $\tau(M) = \Delta_K$, the Alexander polynomial of K.

(5) If $\varphi \in \mathcal{M}_{g,n}$, then $H_*(M(\varphi), \Sigma_+; \mathbb{Z}[H]) = 0$ and therefore $\tau(M(\varphi)) \doteq 1 \in \mathbb{Z}[H]$.

3.3. **Action of homology cylinders on $H_1(\Sigma)$.** Before we can state the behavior of torsion under the product operation we have to study the action of a homology cylinder on the first homology of the surface. Our notation is as follows. Throughout this paper we write $H = H_1(\Sigma)$. Given a homology cylinder (M, i_+, i_-) over Σ we denote the automorphism

$$(i_+)_*^{-1}(i_-)_*\colon H = H_1(\Sigma) \xrightarrow[(i_-)_*]{\cong} H_1(M) \xrightarrow[(i_+)_*^{-1}]{\cong} H$$

by $\varphi(M) = \varphi(M, i_+, i_-)$. Let H_∂ be the image of $H_1(\partial\Sigma)$ in H, and let

$$\mathrm{Aut}^*(H) = \{\varphi \in \mathrm{Aut}(H) \mid \varphi \text{ fixes } H_\partial \text{ and preserves the intersection form of } \Sigma\}.$$

We recall the following result:

Proposition 3.4 (Goda-Sakasai [GS08, Proposition 2.3 and Remark 2.4])**.** *Let M be a homology cylinder over Σ. Then $\varphi(M) \in \mathrm{Aut}^*(H)$.*

Let $\widehat{H} = H/H_\partial = H_1(\widehat{\Sigma})$ where $\widehat{\Sigma}$ denotes Σ with capped boundary circles. We write $H = H_\partial \times \widehat{H}$ by choosing a splitting. Choosing an arbitrary basis of H_∂ and a symplectic basis of $\widehat{H} = H_1(\widehat{\Sigma})$, $\mathrm{Aut}^*(H)$ consists of all matrices P of the form

$$\begin{bmatrix} \mathrm{id}_{n-1} & * \\ 0 & P_0 \end{bmatrix}$$

with $P_0 \in \mathrm{Sp}(2g, \mathbb{Z})$ [GS08, Remark 2.4]. Note that any $P \in \mathrm{Aut}^*(H)$ is indeed realized by a homology cylinder, in fact there exists $\varphi \in \mathcal{M}_{g,n}$ such that the induced action on $H_1(\Sigma_{g,n})$ is given by P (e.g., see [FM09, Proposition 7.3]).

3.4. **Product formulas for torsion.** Each $\varphi \in \mathrm{Aut}(H)$ induces an automorphism of $\mathbb{Z}[H]$, which we also denote by φ. In particular, a homology cylinder M over Σ gives rise to $\varphi(M) \in \mathrm{Aut}(\mathbb{Z}[H])$. We can now formulate the following proposition (e.g., see [Mi66, Section 7] and [Sa08, Proposition 6.6] for related results).

Proposition 3.5. *Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over Σ. Then*

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

Note that $Q(H)^\times$ is a multiplicative abelian group. The action of $\mathrm{Aut}^*(H)$ on H extends to an action of $\mathrm{Aut}^*(H)$ on $Q(H)^\times$, and we can thus form the semidirect product $\mathrm{Aut}^*(H) \ltimes Q(H)^\times$. We can now formulate the following corollary:

Corollary 3.6. *The following is a well-defined homomorphism of monoids*

$$\begin{array}{rcl} \mathcal{C}_{g,n} & \to & \mathrm{Aut}^*(H) \ltimes Q(H)^\times \\ M & \mapsto & (\varphi(M), \tau(M)). \end{array}$$

Since we are mostly interested in abelian quotients of $\mathcal{C}_{g,n}$ we will now show that τ also gives rise to a homomorphism to an abelian group. We define $A(H)$ to be the subgroup of $Q(H)^\times$ generated by the following set

$$\{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H,\, p \in Q(H)^\times, \text{ and } \varphi \in \mathrm{Aut}^*(H)\}.$$

We write $A = A(H)$ when H is clearly understood from the context. The following is an immediate consequence of Proposition 3.5:

Corollary 3.7. *The torsion invariant gives rise to a monoid homomorphism*

$$\tau\colon \mathcal{C}_{g,n} \longrightarrow Q(H)^\times/A.$$

Proof of Proposition 3.5. We write $\Sigma_\pm = i_\pm(\Sigma)$ and $\Lambda_\pm = j_\pm(\Sigma)$. Let $W = M \cup_{\Sigma_- = \Lambda_+} N$, where $i_- \circ (j_+)^{-1}$ is the gluing map. We view M, N, Σ_- ($=\Lambda_+$) as subspaces of W. A Mayer-Vietoris argument shows that the inclusion map induces an isomorphism $H_1(M) \to H_1(W)$. Let $F = H_1(M)$. We equip W with the map $H_1(W) \cong H_1(M) = F$ induced by the inclusion. Restricting this, we equip Σ_- ($= \Lambda_+$), M, N with maps into F. For notational convenience, we denote $Q = Q(F)$, the quotient field of $\mathbb{Z}[F]$. We have the following short exact sequence of cellular homology:

$$0 \longrightarrow C_*(\Sigma_-; Q) \longrightarrow C_*(M, \Sigma_+; Q) \oplus C_*(N; Q) \longrightarrow C_*(W, \Sigma_+; Q) \longrightarrow 0.$$

Here the map $C_*(\Sigma_-; Q) \to C_*(N; Q)$ is given by $-(j_+)_* \circ (i_-)_*^{-1}$ and the map $C_*(\Sigma_-; Q) \to C_*(M, \Sigma_+; Q)$ is given by the inclusion map. Let $\mathcal{B}_*$ be a basis for the graded Q–module $H_*(\Sigma_-; Q)$. It follows from the long exact homology sequence that the inclusion map induces an isomorphism $H_*(\Sigma_-; Q) \to H_*(N; Q)$. We equip $H_*(N; Q)$ with the basis $\mathcal{C}_*$ given by the image of $\mathcal{B}_*$. We consider the resulting torsions $\tau_{\mathcal{B}_*}(\Sigma_-; Q)$, $\tau(M, \Sigma_+; Q)$, $\tau_{\mathcal{C}_*}(N; Q)$, and $\tau(W, \Sigma_+; Q)$. By applying [Mi66, Theorem 3.2] to the above short exact sequence, the torsions satisfy the following:

$$\tau(M, \Sigma_+; Q) \cdot \tau_{\mathcal{C}_*}(N; Q) \doteq \tau(W, \Sigma_+; Q) \cdot \tau_{\mathcal{B}_*}(\Sigma_-; Q).$$

We equip $H_*(\Lambda_+; Q)$ with the basis $\mathcal{D}_* = (j_+)_* \circ (i_-)_*^{-1}(\mathcal{B}_*)$. Similarly we have

$$\tau_{\mathcal{C}_*}(N; Q) \doteq \tau(N, \Lambda_+; Q) \cdot \tau_{\mathcal{D}_*}(\Lambda_+; Q).$$

From the above equations and the tautology $\tau_{\mathcal{D}_*}(\Lambda_+; Q) = \tau_{\mathcal{B}_*}(\Sigma_-; Q)$, it follows that

$$\tau(W, \Sigma_+; Q) \doteq \tau(M, \Sigma_+; Q) \cdot \tau(N, \Lambda_+; Q).$$

We now apply $(i_+)_*^{-1} \colon F \to H$ to the above equality. From the commutative diagram

$$\begin{tikzcd}
H_1(N) \arrow[r, "\cong"] & H_1(W) & H_1(M) = F \arrow[l, "\cong"'] \\
& H \arrow[ul, "(j_+)_*"] \arrow[ur, "(i_-)_*"'] \arrow[r, "\varphi(M)"'] & H \arrow[u, "(i_+)_*"']
\end{tikzcd}$$

it follows that our $H_1(N) \to F$ composed with $(i_+)_*^{-1}$ gives $\varphi(M)(j_+)_*^{-1}$. Therefore we have

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

□

Remark 3.8. Let H be a finitely generated free abelian group. Given a non-zero polynomial $p \in \mathbb{Z}[H]$ we denote by $m(p) \in \mathbb{R}_{>0}$ its Mahler measure (e.g., see [EW99], [SW02], and [SW04]). It is well-known that given $p, q \in \mathbb{Z}[H]$ and $h \in H$ we have $m(p \cdot q) = m(p) \cdot m(q)$ and $m(\pm h \cdot p) = m(p)$. Furthermore given any $\varphi \in \mathrm{Aut}(H)$ we have $m(\varphi(p)) = m(p)$ (e.g., see [EW99, Section 3]). It follows that $M \mapsto m(\tau(M))$ defines a monoid homomorphism $\mathcal{C}_{g,n} \to \mathbb{R}_{>0}$. Using this homomorphism and using the examples of Section 4.3 one can reprove Theorem 2.4 in [GS09].

Now let M be a homology cylinder over a surface $\Sigma = \Sigma_{g,n}$ and let M' be a homology cylinder over a surface $\Sigma' = \Sigma'_{g',n'}$. Assume that $n, n' > 0$ and fix a boundary component c in Σ and fix a boundary component c' in Σ'. Recall that gluing now gives us a homology cylinder $M \cup_{c,c'} M'$ over $\Sigma \cup_{c=c'} \Sigma'$.

Proposition 3.9. *Denote by i and i' the inclusion induced maps of $H_1(\Sigma)$ and $H_1(\Sigma')$ into $H_1(\Sigma \cup_{c=c'} \Sigma')$, respectively. Then the following holds:*

$$\tau(M \cup_{c,c'} M') = i(\tau(M)) \cdot i'(\tau(M')).$$

Proof. Write $H = H_1(\Sigma \cup_{c=c'} \Sigma')$. By the Mayer-Vietoris formula for torsion we have

$$\tau(M \cup_{c,c'} M') = \tau(M; Q(H)) \cdot \tau(M'; Q(H)).$$

It is seen easily that $\tau(M; Q(H)) = i(\tau(M))$ and $\tau(M'; Q(H)) = i'(\tau(M'))$ from the definitions. $\square$

3.5. **Torsion and homology cobordisms.** Let H be a free abelian multiplicative group. We equip $\mathbb{Z}[H]$ with the standard involution of a group ring, i.e. $\overline{h} = h^{-1}$ for $h \in H$ and extend it to $Q(H)$ by setting $\overline{p \cdot q^{-1}} = \overline{p} \cdot \overline{q}^{-1}$.

The next theorem can be viewed as a generalization of the classical Fox-Milnor theorem [FoM66] that the Alexander polynomial of a slice knot factors as $\Delta_K(t) \doteq f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$.

Theorem 3.10. *Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over a surface Σ which are homology cobordant. Write $H = H_1(\Sigma)$. Then*

$$\tau(M) \doteq \tau(N) \cdot q \cdot \bar{q} \in Q(H)^\times$$

for some $q \in Q(H)^\times$.

Proof. Let W be a homology cobordism between M and N. View $\Sigma_+ = i_+(\Sigma) = j_+(\Sigma)$ as a subspace of W. Note that $H_1(\Sigma_+) \to H_1(W)$ is an isomorphism. We now equip W with the homomorphism

$$H_1(W) \xrightarrow{\cong} H_1(\Sigma_+) \xrightarrow{(i_+)_*^{-1}} H.$$

By the above we have $H_1(W, \Sigma_+) = 0$. Therefore it follows from Lemma 3.1 that $H_1(W, \Sigma_+; Q(H)) = 0$ and we can therefore consider $\tau(W) := \tau(W, \Sigma_+; Q(H)) \in Q(H)$. Also note that $H_*(W, M) = 0$ and $H_*(W, N) = 0$. We can hence also think of the torsion invariants $\tau(W, M) := \tau(W, M; Q(H))$ and $\tau(W, N) := \tau(W, N; Q(H))$. From the following short exact sequence of acyclic chain complexes

$$0 \longrightarrow C_*(M, \Sigma_+; Q(H)) \longrightarrow C_*(W, \Sigma_+; Q(H)) \longrightarrow C_*(W, M; Q(H)) \longrightarrow 0,$$

we obtain $\tau(W) \doteq \tau(M) \cdot \tau(W, M)$. Similarly, $\tau(W) \doteq \tau(N) \cdot \tau(W, N)$. By duality we have $\tau(W, M) \doteq \overline{\tau(W, N)}^{-1}$ (e.g., see [Mi66], [KL99] and [CF10]). Hence

$$\tau(M) \doteq \tau(N) \cdot \tau(W, N) \cdot \overline{\tau(W, N)},$$

and the theorem follows. $\square$

Define

$$N(H) = \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^\times,\, h \in H\}.$$

Obviously $N(H)$ is a subgroup of $Q(H)^\times$. As we do with $A = A(H)$, we often write $N = N(H)$.

The following corollary is now an immediate consequence of Corollary 3.6 and Theorem 3.10. This corollary should be compared to [Mo08, Theorem 5.1].

Corollary 3.11. *The map $\mathcal{C}_{g,n} \to \operatorname{Aut}^*(H) \ltimes Q(H)^\times$ defined in Corollary 3.6 descends to a group homomorphism*

$$\mathcal{H}_{g,n} \longrightarrow \operatorname{Aut}^*(H) \ltimes Q(H)^\times / N.$$

Since we are mostly interested in abelian quotients of $\mathcal{H}_{g,n}$ we will for the most part work with the following corollary, which is an immediate consequence of Theorem 3.10.

Corollary 3.12. *The torsion invariant gives rise to a group homomorphism*

$$\tau\colon \mathcal{H}_{g,n} \longrightarrow Q(H)^\times/AN,$$

where $H = H_1(\Sigma_{g,n})$. This descends to a homomorphism of the quotient of $\mathcal{H}_{g,n}$ modulo the normal subgroup $\langle \mathcal{M}_{g,n}\rangle$ generated by the mapping class group $\mathcal{M}_{g,n}$.

In later sections, we will show that the image of this torsion homomorphism is large. Indeed, it has a quotient isomorphic to $(\mathbb{Z}/2)^\infty$ if $b_1(\Sigma_{g,n}) = \operatorname{rank} H > 0$, and has a quotient isomorphic to $\mathbb{Z}^\infty$ if either $n > 2$, or $n = 2$ and $g > 0$.

3.6. **Symmetry and asymmetry of torsion.** It is well-known that the Alexander polynomial $\Delta_K(t)$ of a knot K is symmetric, i.e. $\Delta_K(t) \doteq \Delta_K(t^{-1})$. Furthermore, the Alexander polynomial of any closed 3–manifold or any 3–manifold with toroidal boundary is symmetric (e.g., see [Tu86]). In this section we will study the symmetry properties of the torsion of homology cylinders.

We start out with the following observation.

Lemma 3.13. *Let Σ be a surface which is either an annulus or a surface of genus one with at most one boundary component. Let M be a homology cylinder over Σ. Then $\tau(M) \doteq \overline{\tau(M)}$.*

Proof. First assume that Σ is either an annulus or a torus. We write $H = H_1(M)$, $Q = Q(H)$ and consider the torsion $\tau(M, \Sigma_+; Q)$ corresponding to $\pi_1(M) \to H_1(M) = H$. Clearly it suffices to show that $\tau(M, \Sigma_+; Q)$ is symmetric. First note that Σ_+ has Euler characteristic zero. It can be seen easily that $H_*(\Sigma_+; Q) = 0$, in particular its torsion $\tau(\Sigma_+; Q)$ is defined. Also, from this it follows that M is Q-acyclic so that $\tau(M; Q)$ is defined. It is well-known that the torsion of a torus is trivial and for an annulus it equals $(1-t)^{-1}$, where t is a generator of the first homology group. It follows in particular that $\tau(\Sigma_+; Q) \doteq \tau(\Sigma_-; Q)$. From the long exact sequence of torsion corresponding to the pair $(M, \Sigma_\pm)$ it now follows that $\tau(M, \Sigma_\pm; Q) = \tau(M; Q) \cdot \tau(\Sigma_\pm; Q)^{-1}$. Also, from duality for torsion we have $\tau(M, \Sigma_+; Q) \doteq \overline{\tau(M, \Sigma_-; Q)}$. Combining these equalities, it follows that

$$\begin{aligned}
\tau(M, \Sigma_+; Q) &= \tau(M; Q) \cdot \tau(\Sigma_+; Q)^{-1} \\
&= \tau(M; Q) \cdot \tau(\Sigma_-; Q)^{-1} \\
&= \tau(M, \Sigma_-; Q) \\
&\doteq \overline{\tau(M, \Sigma_+; Q)}
\end{aligned}$$

as desired.

Now assume that Σ is a surface of genus one with exactly one boundary component. Let Σ' be a disk and $T = \Sigma \cup \Sigma'$ the result of gluing Σ and Σ' along their boundary components. Let $M' = \Sigma' \times [0,1]$. Note that $H_1(\Sigma) \to H_1(T)$ is an isomorphism. It now follows immediately from Proposition 3.9 that $\tau(M) = \tau(M \cup M')$, in particular $\tau(M)$ equals the torsion of a homology cylinder over a torus, which is symmetric as we saw above. $\square$

In general torsion is not symmetric though. To our knowledge this phenomenon was first observed in [FJR09, Example 8.5] in the context of sutured manifolds and rational homology cylinders. More examples will be given in Section 4.

On the other hand recall that in order to define the torsion homomorphism in Corollaries 3.7 and 3.12, we thought of torsion invariants modulo $A = A(H)$, i.e., up to the action of $\mathrm{Aut}^*(H)$. When the number of boundary components $n \le 1$, $\mathrm{Aut}^*(H)$ is simply the group of automorphisms on H preserving the intersection pairing on Σ. In particular, the map $\varphi(h) = h^{-1}$ is in $\mathrm{Aut}^*(H)$. Therefore $p \cdot A = \varphi(p) \cdot A = \bar{p} \cdot A$ in $Q(H)^\times/A$ for any p. In other words, modulo A, everything is symmetric:

Lemma 3.14. *Suppose Σ is a surface with at most one boundary components. Then for any homology cylinder M over Σ, we have $\tau(M) = \overline{\tau(M)}$ in $Q(H)^\times/A$.*

Therefore the only remaining case is when either $n > 2$, or $n = 2$ and $g > 0$. We will show that in these cases the torsion invariant $\tau(M)$ is in general asymmetric *even modulo $A = A(H)$*.

In the next section we will consider general methods to construct homology cylinders and to compute their torsion, which will be used to illustrate the asymmetry of torsion.

4. Constructions and computation

In this section we will compute torsion for various homology cylinders. The examples illustrate the computation of torsion and they will also be used later to prove Theorems 1.2 and 1.3.

4.1. **Handle decomposition.** For any homology cylinder (M, i_+, i_-) over Σ, the pair (M, Σ_+) admits a handle decomposition without 0– and 3–handles. A handle decomposition of a pair (M, Σ_+) is given as submanifolds

$$\Sigma_+ \times [0,1] = M_0 \subset M_1 \subset M_2 = M$$

where M_i is obtained by attaching i-handles to M_{i-1} for $i = 1, 2$. M is a homology cylinder if and only if the numbers of 1-handles and 2-handles are equal, say r, and the boundary map

$$\partial \colon C_2(M, \Sigma_+) = \mathbb{Z}^r \longrightarrow C_1(M, \Sigma_+) = \mathbb{Z}^r$$

is invertible.

From the definition, the torsion $\tau(M) = \tau(M, \Sigma_+; Q(H))$ is equal to the determinant of the $r \times r$ matrix A over $\mathbb{Z}[H]$ which represents the $\mathbb{Z}[H]$-coefficient boundary map

$$\partial \colon C_2(M, \Sigma_+; \mathbb{Z}[H]) = \mathbb{Z}[H]^r \longrightarrow C_1(M, \Sigma_+; \mathbb{Z}[H]) = \mathbb{Z}[H]^r$$

with respect to the bases given by 1– and 2–handles. When a handle decomposition is explicitly given, the boundary map can be effectively computed using a standard method.

This is often useful in computing the torsion of a homology cylinder with a given handle decomposition. For readers who are not familiar with this type of computation, it can be described best by a detailed example. The construction and computation below will also be used later, to show the existence of nontrivial homomorphisms of $\mathcal{H}_{g,n}$ onto $\mathbb{Z}$.

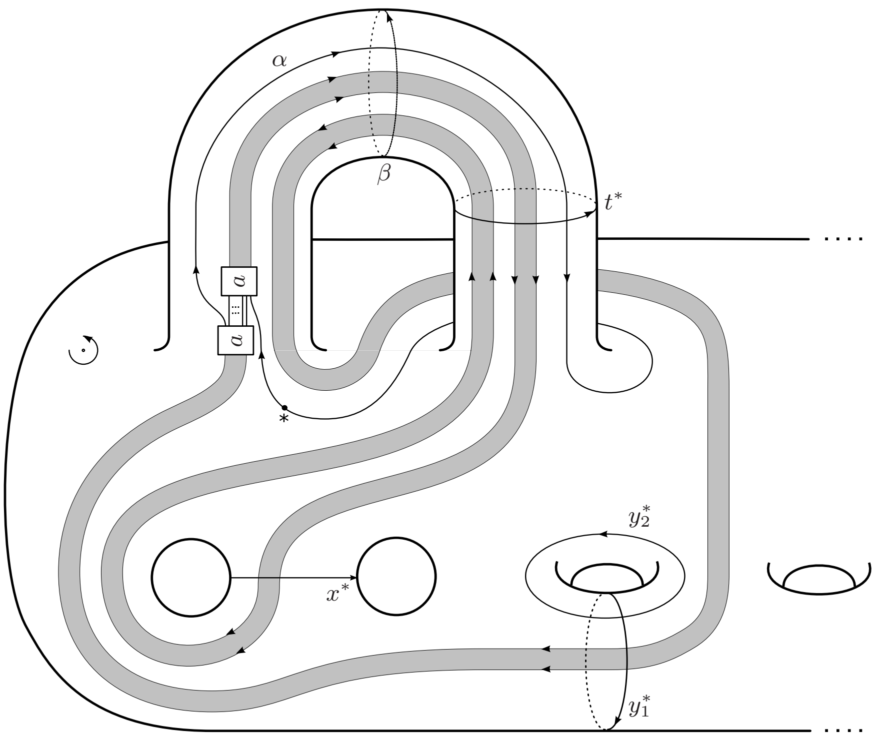

FIGURE 1. Attaching circle α on Σ_1

4.2. **An example of asymmetric torsion.** Let $\Sigma = \Sigma_+ = \Sigma_{g,n}$ with $g > 0$ and $n = 2$. Let M_1 be the 3–manifold obtained by attaching one 1–handle to $\Sigma \times [0, 1]$. The boundary of M_1 is the union of Σ, $\partial\Sigma \times [0, 1]$, and a genus $g + 1$ surface, say Σ_1, which has the same boundary as Σ. We attach one 2–handle to M_1 along the simple closed curve α on Σ_1 as illustrated in Figure 1. The point $*$ is the basepoint and the gray band $\boxed{a}$ — $\boxed{a}$ represents a parallel strands. We denote the resulting 3–manifold by $M = M(a)$.

We have $H_1(M_1) = \mathbb{Z} \times H$ where $\mathbb{Z}$ is generated by the 1–handle. Recall that we can find a splitting $H = H_\partial \times \widehat{H}$ where H_∂ is the image of $H_1(\partial\Sigma)$ and $\widehat{H} = H_1($$\Sigma$ with capped-off boundary). In our case H_∂ and $\widehat{H}$ have rank 1 and $2g$, respectively. We choose generators t of $\mathbb{Z}$, x of H_∂, and y_i $(i = 1, \ldots, 2g)$ of $\widehat{H}$ which lie on Σ_1 and which are dual to the curves t^*, x^*, y_i^* illustrated in Figure 1 with respect to the intersection pairing on Σ_1. That is, t is represented by a loop which is disjoint to x^*, y_i^* and has intersection numbers $+1$ with t^* on Σ_1, and similarly for other generators. Given a curve on Σ_1, its homology class in $H_1(M_1)$ can be determined easily by counting the algebraic intersections with the dual curves.

For the curve α, we have $[\alpha] = (t, x^a y_1^a) \in \mathbb{Z} \times H = H_1(M_1)$. Thus the homology class of α in $H_1(M_1, \Sigma_+) = \mathbb{Z}$ is t, and so the boundary map

$$\partial\colon C_2(M, \Sigma_+) = \mathbb{Z} \longrightarrow C_1(M, \Sigma_+) = \mathbb{Z}$$

is the identity. Therefore our M is a homology cylinder.

Now we compute the $\mathbb{Z}[H]$–coefficient boundary map. Since the attaching circle gives a relation $t = (xy_1)^{-a}$, the generator t is sent to $(xy_1)^{-a}$ under $H_1(M) \xrightarrow{\cong} H_1(\Sigma_+) = H$. The $\mathbb{Z}[H]$–valued intersection number $\lambda(\alpha, \beta)$ of α with the belt circle β of the 1–handle is given by the following formula: for a point u on α, let α_u be the oriented initial segment of α from the basepoint $*$ to u. Then

$$\lambda(\alpha,\beta) = \sum_{u\in\alpha\cap\beta}\left(\epsilon(u)\cdot\left(\prod_{v\in\alpha_u\cap t^*}(xy_1)^{-a\epsilon(v)}\right)\cdot\left(\prod_{c\in\{x,y_1,y_2,\dots,y_{2g}\}}\prod_{w\in\alpha_u\cap c^*}c^{\epsilon(w)}\right)\right).$$

Here, $\epsilon(u)$ is the usual sign of $u \in \alpha \cap \beta$. Precisely, $\epsilon(u) = 1$ if $(d\alpha/dt, d\beta/dt)$ at u agrees with the orientation of Σ_1, and -1 otherwise. The numbers $\epsilon(v)$, $\epsilon(w)$ are given similarly.

From Figure 1, it can be seen that

$$\begin{aligned}\lambda(\alpha,\beta) &= \big(1 + (-1)\cdot x\big) + \big(1\cdot(xy) + (-1)\cdot(x^2y)\big) + \cdots \\ &\quad + \big(1\cdot(x^{a-1}y^{a-1}) + (-1)\cdot(x^a y^{a-1})\big) + 1\cdot x^a y^a \\ &= 1 + (y-1)x + y(y-1)x^2 + \cdots + y^{a-1}(y-1)x^a\end{aligned}$$

where $y = y_1$. The $\mathbb{Z}[H]$–coefficient boundary map is the 1×1 matrix $[\lambda(\alpha, \beta)]$. Therefore $\tau(M(a)) = \lambda(\alpha, \beta)$.

It is obvious that $\tau(M(a))$ is asymmetric in the sense that $\tau(M(a)) \not\doteq \overline{\tau(M(a))}$. In Example 5.9, we will show that $\tau(M(a))$ and $\overline{\tau(M(a))}$ are distinct even modulo $A = A(H)$.

4.3. **Tying in a string link.** In this subsection we describe an operation that modifies a homology cylinder by "tying in" a string link, and investigate its effect on the torsion invariant.

First we recall the definition of a string link. Fix m points $p_1, \dots, p_m$ in the interior of the disk D^2. An m–component string link β is the disjoint union of m properly embedded disjoint oriented arcs β_i $(i = 1, \dots, m)$ in $D^2 \times [0, 1]$ from $p_i \times 0$ to $p_i \times 1$. We denote the exterior of a string link β in $D^2 \times [0, 1]$ by E_β. The string link $\bigcup p_i \times [0, 1] \in D^2 \times [0, 1]$ is called the trivial string link. The exterior of the trivial string link is denoted by E_0. Note that if β is framed, then there is a canonical identification of ∂E_β and ∂E_0 which restricts to the identity on $D^2 \times \{0, 1\} \cap E_0$.

Remark 4.1. Including Theorem 4.2, all results in this section hold for string links in homology $D^2 \times [0, 1]$ as well.

Torsion of string links. We recall the definition of the torsion invariant for string links (cf. [KLW01, Definition 6.8]). Let $X = (D^2 \times 0) \cap E_0$, a subspace of $D^2 \times [0, 1]$. Suppose β is a string link and $\phi\colon H_1(E_\beta) \to H$ is a homomorphism to a free abelian group H. Since (E_β, X) is $\mathbb{Z}$–acyclic, it follows from Lemma 3.1 that (E_β, X) is $\mathbb{Z}[H]$–acyclic. We denote by $\tau^\phi_\beta \in \mathbb{Z}[H]^\times$ the torsion of the cellular chain complex $C_*(E_\beta, X; \mathbb{Z}[H])$. When ϕ is clearly understood from the context we write $\tau_\beta = \tau^\phi_\beta$. As usual, τ_β is well-defined up to multiplication by $\pm h$ $(h \in H)$.

Note that for any string link β, E_β can be viewed as a homology cylinder over the surface X.

Formula for string link tying. Now consider a homology cylinder (M, i_+, i_-) on $\Sigma = \Sigma_{g,n}$ with $H = H_1(\Sigma)$ and an embedding $f\colon E_0 \to \mathrm{int}(M)$. For any framed string link β, define

$$M(f,\beta) = \big(M - f(\mathrm{int}(E_0))\big) \underset{f(\partial E_0)=\partial E_\beta}{\cup} E_\beta.$$

Since E_β has the same homology as E_0, it follows that $(M(f,\beta), i_+, i_-)$ is a homology cylinder. We say that $M(f,\beta)$ is obtained from M by tying the string link β.

The map f gives rise to a homomorphism

$$H_1(E_\beta) \xrightarrow{\cong} H_1(X) \xrightarrow{\cong} H_1(E_0) \xrightarrow{f_*} H_1(M) \longleftarrow H_1(\Sigma_+) \xrightarrow{(i_+)_*^{-1}} H_1(\Sigma) = H$$

induced by the inclusions. We denote the resulting torsion invariant of β by τ^f_β.

Theorem 4.2. *The torsion invariant of the homology cylinder $M(f,\beta)$ is given by $\tau(M(f,\beta)) \doteq \tau(M)\cdot\tau^f_\beta$.*

Proof. We write $\Sigma_+ = i_+(\Sigma)$. Let $M' = M - f(\mathrm{int}(E_0))$. Choose bases of $H_*(X;Q(H))$ and $H_*(M',\Sigma_+;Q(H))$. Since (E_β, X) is $\mathbb{Z}$–acyclic, it is $Q(H)$–acyclic, and therefore for any string link β (including the trivial one) our basis of $H_*(X;Q(H))$ determines a basis of $H_*(E_\beta;Q(H))$ via the inclusion-induced map. Define the torsion invariants $\tau(X)$, $\tau(E_\beta)$, $\tau(M',\Sigma_+) \in Q(H)^\times$ (up to the usual ambiguity) using these bases.

Consider the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow C_*(\partial E_0;Q(H)) \longrightarrow C_*(M',\Sigma_+;Q(H)) \oplus C_*(E_\beta;Q(H)) \\ \longrightarrow C_*(M(f,\beta),\Sigma_+;Q(H)) \longrightarrow 0. \end{aligned}$$

Since $(M(f,\beta),\Sigma_+)$ is $Q(H)$-acyclic, the bases we have chosen give rise to a basis of $H_*(\partial E_0;Q(H))$. Define $\tau(\partial E_0) \in Q(H)^\times$ using this basis. By [Mi66, Theorem 3.2], we have

$$\tau(M',\Sigma_+)\cdot\tau(E_\beta) \doteq \tau(\partial E_0)\cdot\tau(M(f,\beta)).$$

Considering the special case of a trivial string link β, we obtain

$$\tau(M',\Sigma_+)\cdot\tau(E_0) \doteq \tau(\partial E_0)\cdot\tau(M).$$

Combining these, we have

$$\tau(M(f,\beta)) \doteq \tau(M)\cdot\tau(E_\beta)\cdot\tau(E_0)^{-1}.$$

From the short exact sequence

$$0 \longrightarrow C_*(X;Q(H)) \longrightarrow C_*(E_\beta;Q(H)) \longrightarrow C_*(E_\beta,X;Q(H)) \longrightarrow 0$$

with (E_β, X) acyclic, we obtain $\tau(E_\beta) \doteq \tau(X)\cdot\tau^f_\beta$. Note that $E_0 = X\times[0,1]$, hence $\tau(E_0) = \tau(X)$. The desired formula now follows immediately. ☐

Special case: tying in a knot. We will now study the case $m = 1$. Note that a string link β with one component gives us a knot K in S^3. Furthermore, for any embedding $f\colon E_0 \to \mathrm{int}(M)$, it can be seen easily that the torsion τ^f_β is equal to $\Delta_K(h)$, where $\Delta_K(t)$ is the Alexander polynomial of K and h is the image of the generator of $H_1(E_0) = \mathbb{Z}$ under the map $H_1(E_0) \to H$ induced by f. Therefore, by Theorem 4.2, the torsion of $M(f,\beta)$ can be described in terms of the Alexander polynomial of K. The following is a special case of this, which gives a group

homomorphism of the (smooth or topological) concordance group $\mathcal{C}$ of knots in S^3 into $\mathcal{H}_{g,n}$. Recall that $E_0 = X \times [0,1]$ and $X = E_0 \cap (D^2 \times 0)$.

Proposition 4.3. *Let $g, n \geq 0$. We write $\Sigma = \Sigma_{g,n}$ and $M = \Sigma \times [0,1]$. Let $\iota\colon X \to \operatorname{int}(\Sigma)$ be an embedding, and let $f\colon E_0 \to \operatorname{int}(M)$ be the embedding $f(x,t) = \iota(x, t/2+1/4)$. Then the assignment $K \mapsto M(f,K)$ descends to a group homomorphism*

$$\mathcal{C} \longrightarrow \mathcal{H}_{g,n}.$$

Furthermore

$$\tau(M(f,K)) = \Delta_K(h)$$

where h is the image of the generator of $H_1(X) \cong \mathbb{Z}$ under $\iota_*\colon H_1(X) \to H$.

Proof. It is straightforward to verify that $M(f, K_1 \# K_2) \cong M(f,K_1) \cdot M(f,K_2)$ and that the assignment $K \mapsto M(f,K)$ in fact descends to a group homomorphism $\mathcal{C} \to \mathcal{H}_{g,n}$. The conclusion on torsion has already been proven in the paragraph above Proposition 4.3. $\square$

5. Epimorphisms onto infinitely generated abelian groups

In this section we will construct epimorphisms of $\mathcal{H}_{g,n}$ onto nontrivial abelian groups. This will give a proof of Theorems 1.2 and 1.3. Throughout Section 5, we fix $g, n \geq 0$ and write $\Sigma = \Sigma_{g,n}$ and $H = H_1(\Sigma)$.

5.1. **Algebraic structure of the torsion group.** Recall that our torsion invariant lives in the multiplicative abelian group $Q(H)^\times/AN$. In this subsection we investigate the algebraic structure of $Q(H)^\times/AN$. We will think of symmetric/asymmetric parts of $Q(H)^\times/AN$, and define certain $\mathbb{Z}/2$ and $\mathbb{Z}$-valued homomorphisms which form sets of complete invariants of these parts, respectively. To be more precise, let

$$Q(H)^{sym} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times/A\}.$$

Note that $AN \subset Q(H)^{sym}$. There is an exact sequence

$$1 \longrightarrow \frac{Q(H)^{sym}}{AN} \longrightarrow \frac{Q(H)^\times}{AN} \longrightarrow \frac{Q(H)^\times}{Q(H)^{sym}} \longrightarrow 1$$

which can be viewed as a decomposition of $Q(H)^\times/AN$ into symmetric and asymmetric parts.

Recall that H_∂ is the image of $H_1(\partial\Sigma)$ in H and

$$\operatorname{Aut}^*(H) = \{\varphi \in \operatorname{Aut}(H) \mid \varphi \text{ fixes } H_\partial \text{ and preserves the intersection form of } \Sigma\}.$$

We define an equivalence relation $\sim$ on $\mathbb{Z}[H] - \{0\}$ by $p \sim q$ if $p \doteq \varphi(q)$ for some $\varphi \in \operatorname{Aut}^*(H)$. Note that if $p \sim q$ then p and q represent the same element in $Q(H)^\times/A$. From now on we say p is *self-dual* if $p \sim \bar{p}$.

Recall that $\mathbb{Z}[H]$ is a unique factorization domain, so that for each $p \in Q(H)$ and irreducible $\lambda \in \mathbb{Z}[H]$, we can think of the exponent of λ in the factorization of p. (The exponent is an integer and may be negative.) For an irreducible element λ in $\mathbb{Z}[H]$, we define a function $e_\lambda\colon Q(H)^\times \to \mathbb{Z}$ as follows: Given $p \in Q(H)$, e_λ is the sum of exponents of distinct irreducible factors μ of p such that $\mu \sim \lambda$. (As usual, irreducible factors are distinguished up to multiplication by a unit in $\mathbb{Z}[H]$.)

Proposition 5.1.

(1) *If λ is a self-dual irreducible element in $\mathbb{Z}[H]$, then the map*

$$\Psi_\lambda \colon Q(H)^{sym}/AN \longrightarrow \mathbb{Z}/2$$

defined by $\Psi_\lambda(p\cdot AN) = e_\lambda(p) + 2\mathbb{Z}$ is a surjective group homomorphism. Furthermore,

$$\Psi = \bigoplus_{[\lambda]} \Psi_\lambda \colon Q(H)^{sym}/AN \longrightarrow \bigoplus_{[\lambda]} \mathbb{Z}/2,$$

is an isomorphism, where $[\lambda]$ runs over the equivalence classes of self-dual irreducible λ.

(2) *If μ is a non-self-dual irreducible element in $\mathbb{Z}[H]$, then the map*

$$\Theta_\mu \colon Q(H)^\times/Q(H)^{sym} \longrightarrow \mathbb{Z}$$

defined by $\Theta_\mu(p\cdot Q(H)^{sym}) = e_\mu(p) - e_{\bar\mu}(p)$ is a surjective group homomorphism. Furthermore

$$\Theta = \bigoplus_{\{[\mu],[\bar\mu]\}} \Theta_\mu \colon Q(H)^\times/Q(H)^{sym} \longrightarrow \bigoplus_{\{[\mu],[\bar\mu]\}} \mathbb{Z},$$

is an isomorphism, where $\{[\mu],[\bar\mu]\}$ runs over the unordered pairs of equivalence classes of non-self-dual irreducible μ and its involution $\bar\mu$.

Consequently, $Q(H)^\times/AN$ is isomorphic to $\big(\bigoplus_{[\lambda]} \mathbb{Z}/2\big) \oplus \big(\bigoplus_{\{[\mu],[\bar\mu]\}} \mathbb{Z}\big)$.

Remark 5.2.

(1) In the definition of Θ, we have one summand for the two classes $[\mu]$ and $[\bar\mu]$. Here we have sign ambiguity since $\Theta_\mu = -\Theta_{\bar\mu}$, but this does not cause any problems in our conclusions.

(2) The homomorphisms Ψ_λ and Ψ extend to homomorphisms of $Q(H)^\times/AN$, which will also be denoted by Ψ_λ and Ψ. Also, we denote by Θ and Θ_μ the homomorphisms of $Q(H)^\times/AN$ induced by Θ and Θ_μ. Then the isomorphism in the last sentence of Theorem 5.3 can be written as (Ψ,Θ).

(3) Although Θ_μ could be defined for self-dual μ as well, it is not interesting since the resulting Θ_μ is always zero.

Proof. First we will observe that Ψ_λ and Ψ are well-defined surjective homomorphisms. Since the factorization into irreducible factors is preserved by the $\mathrm{Aut}^*(H)$-action, e_λ is invariant under $\sim$ for any irreducible λ. If λ is self-dual, then $e_\lambda(u\bar u) = e_\lambda(u) + e_{\bar\lambda}(\bar u) = 2e_\lambda(u)$. It follows that Ψ_λ is a well-defined homomorphism. The surjectivity of Ψ_λ and Ψ follows from the observation that $\Psi_{\lambda'}(\lambda\cdot A)$ equals 1 if $\lambda' \sim \lambda$, and 0 otherwise.

To see that Θ_μ and Θ are well-defined homomorphisms, observe that for $p \in Q(H)^{sym}$, $e_\mu(p) = e_{\bar\mu}(\bar p) = e_{\bar\mu}(p)$. Their surjectivity now follows from the observation that for non-self-dual irreducible μ and μ', $\Theta_{\mu'}(\mu\cdot Q(H)^{sym})$ equals 1 if $\mu' \sim \mu$, -1 if $\mu' \sim \bar\mu$, and 0 otherwise.

For the injectivity of Ψ, suppose $f \in Q(H)^{sym}$ represents an element in the kernel of Ψ. We can rewrite the irreducible factorization of f to obtain an expression $f = \lambda_1^{m_1}\cdots\lambda_r^{m_r}\cdot u$, where the λ_i are mutually non-equivalent self-dual irreducible elements, $m_i \in \mathbb{Z}$, and $u \in AN$. Note that $\lambda_i^2 = \lambda_i\bar\lambda_i = 1$ in $Q(H)^{sym}/AN$ since λ_i is self-dual. Evaluating Ψ_{λ_i}, we have that m_i is even for each i. From this it follows that $f = 1$ in $Q(H)^{sym}/AN$.

The injectivity of Θ is proved similarly: if $f \in Q(H)^\times$ represents an element in the kernel of Ψ, then from the irreducible factorization of f we obtain an expression $f = \mu_1^{m_1}\bar{\mu}_1^{n_1}\cdots\mu_r^{m_r}\bar{\mu}_r^{n_r}\cdot u$, where $m_i, n_i \in \mathbb{Z}$, $u \in Q(H)^{sym}$, and the μ_i are non-self-dual irreducible elements such that $\mu_i \not\sim \mu_j \not\sim \bar{\mu}_i$ whenever $i \neq j$. Evaluating Θ_{μ_i}, we have $m_i = n_i$ for each i. It follows that $f = 1$ in $Q(H)^\times/Q(H)^{sym}$. □

Now, in order to understand the structure of $Q(H)^\times/AN$, the only remaining part is to count the number of summands of Ψ and Θ in Proposition 5.1. To state the result, we introduce the following definition: we say that (g,n) is *small* if either $n \le 1$ or $n = 2$ and $g = 0$. Otherwise we say that (g,n) is *large*.

Theorem 5.3. *Suppose H is nontrivial, i.e., $\Sigma_{g,n}$ is neither a sphere nor disk. Then the following hold:*

1. *Ψ is an isomorphism of $Q(H)^{sym}/AN$ onto $(\mathbb{Z}/2)^\infty$.*
2. *If (g,n) is small, then $Q(H)^\times/Q(H)^{sym} = 0$. If (g,n) is large, then Θ is an isomorphism of $Q(H)^\times/Q(H)^{sym}$ onto $\mathbb{Z}^\infty$.*

Consequently,

$$\frac{Q(H)^\times}{AN} \cong \begin{cases} (\mathbb{Z}/2)^\infty & \text{if } (g,n) \text{ is small,} \\ (\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty & \text{if } (g,n) \text{ is large.} \end{cases}$$

Proof. Recall that in Section 3.6 we observed that there is no non-self-dual μ if (g,n) is small. The first sentence of Theorem 5.3 (2) is an immediate consequence. In the following subsections, we will realize, as the values of torsion invariants of homology cylinders, infinitely many self-dual classes $[\lambda]$ when H is nontrivial (see Theorem 5.4), and infinitely many non-self-dual classes $[\mu]$ when (g,n) is large (see Theorem 5.6). □

5.2. **Proofs of Theorems 1.2 and 1.3.** In Sections 5.2 and 5.3, we give proofs of Theorems 1.2 and 1.3. Along the way we also conclude the proof of Theorem 5.3.

Theorem 5.4. *If $b_1(\Sigma_{g,n}) > 0$, then there exists a subgroup $\mathcal{S} \subset \mathcal{H}_{g,n}$ isomorphic to $(\mathbb{Z}/2)^\infty$ such that*

$$\mathcal{S} \longrightarrow \mathcal{H}_{g,n} \xrightarrow{\tau} Q(H)^\times/AN \xrightarrow{\Psi} \bigoplus_{[\lambda]} \mathbb{Z}/2$$

is an injection whose image is the sum of a certain infinite set of $\mathbb{Z}/2$ summands of $\bigoplus_{[\lambda]} \mathbb{Z}/2$.

Proof. We pick non-trivial knots K_i $(i = 1, 2\ldots)$ which are negative amphicheiral with irreducible Alexander polynomials such that the multisets C_i of nonzero coefficients of $\Delta_i(t) := \Delta_{K_i}(t)$ are mutually distinct up to sign. For example, one can use the family of knots described in [Ch07, p. 60]: their Alexander polynomials are of the form $a^2t^2 - (2a^2+1)t + a^2$. It is well-known that the knots K_i form a $(\mathbb{Z}/2)$-basis of a subgroup of the knot concordance group isomorphic to $(\mathbb{Z}/2)^\infty$.

We write $M = \Sigma \times [0,1]$. Let $f\colon E_0 \to \operatorname{int}(M)$ be an embedding as in Proposition 4.3, which is homologically essential. Denote by h the image of the generator of $H_1(E_0) \cong \mathbb{Z}$ under the homomorphism $H_1(E_0) \to H$ induced by the given embedding $f\colon E_0 \to M$.

We now write $M_i = M(f, K_i)$. By Proposition 4.3, the M_i span a subgroup of $\mathcal{H}_{g,n}$ which is the homomorphic image of $(\mathbb{Z}/2)^\infty$. Furthermore we have $\tau(M_i) =$

$\Delta_i(h)$. Note that each $\Delta_i(h)$ is irreducible and self-dual since $\Delta_i(t)$ is irreducible and self-dual and since h is easily seen to be an indivisible element in H. It is not difficult to see directly that the multiset C_i is an invariant of $\Delta_i(h)$ under the equivalence relation $\sim$. (For a more general method from which this observation is derived as a special case, see Section 5.3.) Therefore, since the C_i are all distinct, the equivalence classes of the $\Delta_i(h)$ are mutually distinct. From this we obtain

$$\Psi_{\Delta_i(h)}(\tau(M_j)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\Psi_\lambda(\tau(M_i)) = 0$ if $\lambda \not\sim \Delta_i(h)$. Therefore the composition $\mathcal{S} \to \bigoplus_{[\lambda]} \mathbb{Z}/2$ in the statement of this theorem is injective and has image $\bigoplus_{[\Delta_i(h)]} \mathbb{Z}/2 \cong (\mathbb{Z}/2)^\infty$. $\square$

We now obtain Theorem 1.2 as an immediate corollary.

Proof of Theorem 1.2. By Theorem 5.4, we have a subgroup $\mathcal{S}$ of $\mathcal{H}_{g,n}$ and a homomorphism $\mathcal{H}_{g,n} \to (\mathbb{Z}/2)^\infty$ whose restriction to $\mathcal{S}$ is an isomorphism. It follows that the homomorphism splits, and $\mathcal{S} \cong (\mathbb{Z}/2)^\infty$ descends to a summand of the abelianization of $\mathcal{H}_{g,n}$. $\square$

Remark 5.5.

(1) Using the full power of Theorem 4.2, i.e., by tying in string links with several components, we can realize many more nontrivial values of the homomorphism Ψ.

(2) Following the arguments in the proof of Theorem 1.2, one can easily show that if $b_1(\Sigma) > 0$ then there exists a commutative diagram

$$\begin{tikzcd} (\mathbb{Z}/2)^\infty \arrow[r, "\mathrm{id}"] \arrow[d] & (\mathbb{Z}/2)^\infty \\ \mathcal{H}_{g,n}^{\mathrm{smooth}} \arrow[r] & \mathcal{H}_{g,n}^{\mathrm{top}} \arrow[u] \end{tikzcd}$$

such that the left hand map is injective and the right hand map is surjective.

We also have the following realization result:

Theorem 5.6. *If (g, n) is large, then the image of*

$$\mathcal{H}_{g,n} \xrightarrow{\tau} Q(H)^\times/AN \xrightarrow{\Theta} \bigoplus_{\{[\mu],[\bar{\mu}]\}} \mathbb{Z}$$

contains infinitely many summands of $\bigoplus_{\{[\mu],[\bar{\mu}]\}} \mathbb{Z}$.

The proof of Theorem 5.6 requires a more sophisticated method to detect non-equivalence of non-self-dual irreducible factors. This will occupy all of Section 5.3. Assuming Theorem 5.6 we can now finally prove Theorem 1.3:

Proof of Theorem 1.3. Suppose that $n > 1$. By Theorem 2.5, we may assume that (g, n) is large. Let $\mathcal{H}_{g,n}^{ab}$ be the abelianization of $\mathcal{H}_{g,n}$. By Theorem 5.6, we have a surjection $\mathcal{H}_{g,n} \to \mathbb{Z}^\infty$. This induces a split surjection, say g, of the abelianization $\mathcal{H}_{g,n}^{ab}$ of $\mathcal{H}_{g,n}$ onto $\mathbb{Z}^\infty$. Also, by Theorem 5.4, we have a split surjection $f\colon \mathcal{H}_{g,n}^{ab} \to (\mathbb{Z}/2)^\infty$. Since the intersection of the images of the right inverses of f and g is automatically $\{0\}$ (e.g., compare the order), it follows that $(f, g)\colon \mathcal{H}_{g,n}^{ab} \to (\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$ is a split surjection. $\square$

5.3. **Detecting non-equivalent non-self-dual factors.** In order to prove Theorem 5.6 we first will introduce a simple and practical method for distinguishing elements in $\mathbb{Z}[H]$ up to the action of $\mathrm{Aut}^*(H)$.

As before, we denote $H = H_1(\Sigma)$ and $H_\partial = \mathrm{Im}\{H_1(\partial\Sigma) \to H\}$. Denote $\widehat{H} = H/H_\partial = H_1(\widehat{\Sigma})$ where $\widehat{\Sigma}$ is Σ with boundary circles capped off, and write $H = H_\partial \times \widehat{H}$ by choosing a splitting. Fix a basis $\{x_1, \dots, x_{n-1}\}$ of H_∂, so that each $u \in \mathbb{Z}[H_\partial]$ is viewed as a (Laurent) polynomial in the variables x_i. For $u, v \in \mathbb{Z}[H_\partial]$, we write $u \approx v$ if $u = v \cdot m$ for some $m \in H_\partial$. This is an equivalence relation; denote the equivalence class of $u \in \mathbb{Z}[H_\partial]$ by $[u]$. Since $u \approx v$ if and only if the polynomial u is obtained from v by shifting the exponents, it is very straightforward to check whether $u \approx v$ or not for two given polynomials u and v. Given $p \in \mathbb{Z}[H]$, write $p = \sum_{g \in \widehat{H}} u_g \cdot g$ where $u_g \in \mathbb{Z}[H_\partial]$, and define

$$C(p) = \{[u_g] \mid g \in \widehat{H}\}.$$

We view $C(p)$ as a multiset, i.e., repeated elements are allowed. For $C(p) = \{[u_g]\}$, denote $-C(p) = \{[-u_g]\}$.

Lemma 5.7. *$C(p)$ is invariant up to sign under $\sim$ on $\mathbb{Z}[H]$, i.e., if $p \sim q$, $C(p)$ is equal to either $C(q)$ or $-C(q)$.*

Proof. Note that $\varphi \in \mathrm{Aut}^*(H)$ fixes H_∂ and sends $g \in \widehat{H}$ to an element of the form $g' m_g$ for some $g' \in \widehat{H}$, $m_g \in H_\partial$. In addition the association $g \mapsto g'$ is a bijection since φ is an isomorphism. Therefore, for $p = \sum_{g \in \widehat{H}} u_g \cdot g \in \mathbb{Z}[H]$, the classes $[u_g]$ are permuted by the action of φ. It follows $C(p) = C(\varphi(p))$. It is easily seen that $C(\pm p \cdot h) = \pm C(p)$ for $h \in H$. $\square$

Example 5.8. In the proof of Theorem 5.4, we have observed that for $p = \sum_{h \in H} c_h \cdot h \in \mathbb{Z}[H]$, the multiset of all nonzero coefficients $\{c_h \mid h \in H \text{ and } c_h \neq 0\}$ is an invariant, up to sign, of p under $\sim$. This can be viewed as a consequence of Lemma 5.7, since the multiset of nonzero coefficients of an element $u_g \in \mathbb{Z}[H_\partial]$ is invariant under $\approx$.

We remark that if $C(p) = \{[u_g]\}$, then $C(\bar{p}) = \{[\bar{u}_g]\}$. This combined with Lemma 5.7 often allows us to detect non-self-dual elements, as illustrated below.

Example 5.9. Fix $g \neq e \in \widehat{H}$. For a positive integer a, let

$$p_a = 1 + (g-1)x_i + g(g-1)x_i^2 + \cdots + g^{a-1}(g-1)x_i^a.$$

Then

$$\begin{aligned} C(p_a) &= \{[1 - x_i], \dots, [x_i^{a-1} - x_i^a], [x_i^a]\} = \{[1 - x_i], \dots, [1 - x_i], [1]\}, \\ C(\bar{p}_a) &= \{[x_i - 1], \dots, [x_i - 1], [1]\}. \end{aligned}$$

Looking at the element $[1]$ we see that $C(p_a) \neq -C(p_a)$. Since $1 - x_i \not\approx x_i - 1$ we deduce that $C(p_a) \neq C(\bar{p}_a)$. Therefore $p_a \not\sim \bar{p}_a$, i.e., p_a is non-self-dual. In particular, the torsion of the homology cylinder $M(a)$ in Section 4.2 is non-self-dual. Also, since $|C(p_a)| = |C(\bar{p}_a)| = a + 1$, we have $p_a \not\sim p_b \not\sim \bar{p}_a$ whenever $a \neq b$.

We are now finally in a position to prove Theorem 5.6.

Proof of Theorem 5.6. First we consider the case $n = 2$ and $g > 0$. For each positive integer a, consider the homology cylinder $M(a)$ constructed in Section 4.2. Let $p_a = \tau(M(a)) \in \mathbb{Z}[H]$. As we observed in Example 5.9, p_a is non-self-dual, and $p_a \not\sim p_b \not\sim \bar{p}_a$ whenever $a \neq b$. Applying the Eisenstein criterion to p_a, it can be seen that p_a is irreducible. Therefore

$$\Theta_{p_a}(\tau(M(b))) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that the image of the subgroup generated by the classes of the $M(a)$ under the homomorphism

$$\mathcal{H}_{g,n} \xrightarrow{\tau} Q(H)^\times/AN \xrightarrow{\Theta} \bigoplus_{\{[\mu],[\bar{\mu}]\}} \mathbb{Z}$$

is equal to $\bigoplus_{\{[p_a],[\bar{p}_a]\}} \mathbb{Z} \cong \mathbb{Z}^\infty$. This proves the theorem for this case.

For $n \geq 3$, the theorem is proved by a similar construction of homology cylinders, using the generator of H associated to a third boundary component of Σ in place of the generator y_1. (In this case the torsion p_a lies in $\mathbb{Z}[H_\partial]$ and thus it is easier to see that $p_a \not\sim p_b \not\sim \bar{p}_a$ whenever $a \neq b$.) $\square$

6. Pretzel links

In this section we study homology cylinders arising from pretzel links. The pretzel link $P(2r, 2s, 2t)$ is a 3–component link with an obvious Seifert surface Σ, as shown in Figure 2. We pick Σ as a model surface for $\Sigma_{0,3}$. We then write $M(r,s,t) = (S^3 \text{ cut along } \Sigma, i_+, i_-)$.

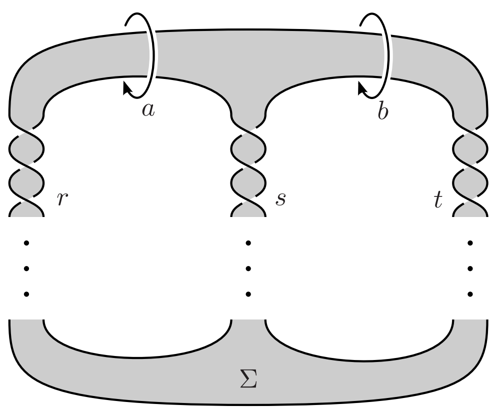

Figure 2. The pretzel link $P(2r, 2s, 2t)$ with r, s, and t full twists.

The main result of this section is the following.

Proposition 6.1.

(1) *$M(r,s,t)$ defines a homology cylinder if and only if $(r+s)(t+s) - s^2 = \pm 1$.*

(2) *The set of homology cylinders $M(r,s,t)$ generates a $\mathbb{Z}^\infty$ subsummand of the abelianization of $\mathcal{H}_{0,3}$.*

The proof of the proposition will require the remainder of this section. Given r, s, t, we write $M = S^3$ cut along Σ. Note that M is a handlebody and that $\pi_1(M)$ is the free group on the generators a and b as shown in Figure 2.

Let α be a loop on Σ which runs from down the left-hand strip, and back up via the middle strip. Similarly, let β be a loop which runs down the right-hand strip and back up the middle, so that $\pi_1(\Sigma)$ is generated by α and β. We now denote by α^+ and β^+ the corresponding curves on $\Sigma^+ \subset M$. Then

$$\alpha^+ = a^r(ab^{-1})^s \qquad \beta^+ = b^{-t}(ab^{-1})^s.$$

We find that $H_1(\Sigma) \to H_1(M)$ is an isomorphism if and only if $(r+s)(t+s)-s^2 = \pm 1$. This concludes the proof of Proposition 6.1 (1).

We will now calculate the torsions for homology cylinders with $(r+s)(t+s)-s^2 = 1$. Note that this condition on r, s, t is equivalent to $(r+s)(t+s) = s^2+1$. In particular one of r, s or t is necessarily negative.

Using the symmetries of Pretzel links we can without loss of generality assume that $r, s > 0$ and $t < 0$. Note that the condition $s^2+1 = (r+s)(t+s)$ implies that $|t| < s$ and $|t| < r$.

Recall that we can view $H_1(M(r,s,t))$ as the free abelian multiplicative group with basis $\{a, b\}$. Using the isomorphism $H_1(\Sigma_{0,3}) \to H_1(\Sigma^+) \to H_1(M(r,s,t))$ we now identify $H_1(\Sigma_{0,3})$ with the multiplicative free abelian group generated by a and b. In particular we will view $\tau(M(r,s,t))$ as an element in $\mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$.

Evaluating the 2×2 determinant

$$\begin{bmatrix} \partial\alpha^+/\partial a & \partial\beta^+/\partial a \\ \partial\alpha^+/\partial b & \partial\beta^+/\partial b \end{bmatrix}$$

we obtain from [FJR09, Proposition 4.2] that $\tau(M(r,s,t))$ equals

$$(*) \qquad \begin{array}{rl} & (1+a+\dots+a^{r-1})(1+b+\dots+b^{|t|-1}) \\ + & a^r(1+b+\dots+b^{|t|-1})(1+ab^{-1}+\dots+(ab^{-1})^{s-1}) \\ - & ab^{|t|-1}(1+a+\dots+a^{r-1})(1+ab^{-1}+\dots+(ab^{-1})^{s-1}). \end{array}$$

This polynomial is always asymmetric. In fact the support of the polynomial $\tau(M(r,s,t))$ is given in Figure 3. Here a point (i,j) corresponds to a^ib^j. Furthermore a triangle (▲) indicates a coefficient -1 and a big five-pointed star (★) indicates a coefficient $+1$.

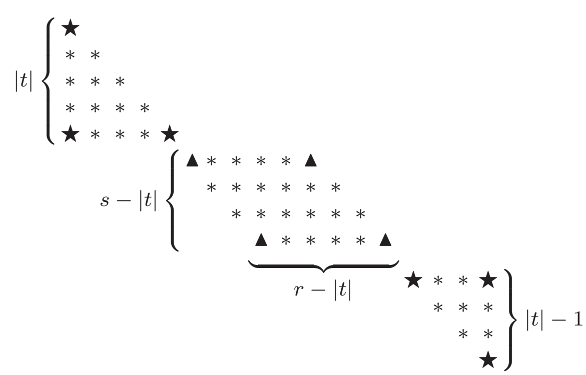

FIGURE 3. Support of $\tau(M(r,s,t))$.

Proposition 6.2. *There exist infinitely many positive integers r_i, s_i, negative integers t_i, and irreducible polynomials $p_i \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$ with the following properties:*

(1) $s_i^2 + 1 = (r_i + s_i)(t_i + s_i)$,
(2) $p_i \not\doteq \overline{p_i}$, *and*
(3) p_i *divides* $\tau(M(r_j, s_j, t_j))$ *if and only if* $i = j$.

Proof. Pick distinct odd primes $x_1, x_2, \dots$. Furthermore pick $r_i > x_i$ and $s_i > x_i$ such that $(r_i - x_i)(s_i - x_i) = 1 + x_i^2$. We then have

$$\begin{aligned}(r_i + s_i)(s_i - x_i) &= ((r_i - x_i) + (s_i + x_i))(s_i - x_i)\\ &= (s_i + x_i)(s_i - x_i) + (r_i - x_i)(s_i - x_i)\\ &= s_i^2 - x_i^2 + 1 + x_i^2\\ &= s_i^2 + 1.\end{aligned}$$

We now set $t_i = -x_i$, it follows that $s_i^2 + 1 = (r_i + s_i)(t_i + s_i)$.

In order to show the existence of p_i with the required properties we have to introduce various definitions. Given $p \in \mathbb{Z}[b^{\pm 1}][a^{\pm 1}]$ we now denote by $l(p) \in \mathbb{Z}[b^{\pm 1}]$ the coefficient of the lowest degree and by $h(p) \in \mathbb{Z}[b^{\pm 1}]$ the coefficient of the highest degree. Note that for $p, q \in \mathbb{Z}[b^{\pm 1}][a^{\pm 1}]$ we have $l(p \cdot q) = l(p) \cdot l(q)$ and $h(p \cdot q) = h(p) \cdot h(q)$.

We now write $\tau_i = \tau(M(r_i, s_i, t_i))$ and we view τ_i as an element in $\mathbb{Z}[b^{\pm 1}][a^{\pm 1}]$. By Equation $(*)$ we have

$$l(\tau_i) = 1 + b + b^2 + \dots + b^{x_i - 1}.$$

Since x_i is prime it follows that $l(\tau_i) \in \mathbb{Z}[b^{\pm 1}]$ is irreducible. In particular for any i there exists an irreducible factor $p_i \in \mathbb{Z}[b^{\pm 1}][a^{\pm 1}]$ of τ_i with $l(p_i) = 1 + b + b^2 + \dots + b^{x_i - 1}$.

Note that $h(p_i)$ divides $h(\tau_i) = 1 + b + \dots + b^{x_i - 2}$. In particular $\deg(h(p_i)) = x_i - 2$ and $\deg(l(p_i)) = x_i - 1$. It follows easily that $p_i \not\doteq \overline{p_i}$.

Given i, j we have $l(p_i) = 1 + b + \dots + b^{x_i - 1}$ and $l(\tau_j) = 1 + b + \dots + b^{x_j - 1}$. In particular if $i \neq j$, then $l(p_i)$ and $l(\tau_j)$ are distinct irreducible polynomials. It follows that p_i does not divide τ_j if $i \neq j$. $\square$

Write $H = H_1(\Sigma_{0,3})$. Note that $\operatorname{Aut}^*(H) = \{\mathrm{id}\}$. In particular the polynomials $p_i \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}] = \mathbb{Z}[H]$ satisfy $p_i \not\sim \overline{p_i}$. It now follows immediately that the homology cylinders $M(r, s, t)$ span a subgroup of $\mathcal{H}_{0,3}$ which surjects onto $\mathbb{Z}^\infty$ under the map $\Theta \circ \tau$. This concludes the proof of Proposition 6.1.

7. The Torelli subgroup

In this section, we consider a subgroup of $\mathcal{H}_{g,n}$ which generalizes the Torelli group of the mapping class group. We prove analogues of Theorems 1.2 and 1.3, but for a larger set of surfaces.

Let $g, n \geq 0$ and write $H = H_1(\Sigma_{g,n})$. Recall that each $\varphi \in \mathcal{M}_{g,n}$ induces an action φ_* on H and the map $\mathcal{M}_{g,n} \to \operatorname{Aut}^*(H)$ sending φ to φ_* is an epimorphism. The Torelli group $\mathcal{I}_{g,n}$ is defined to be the kernel of this map, i.e. $\mathcal{I}_{g,n}$ is the subgroup of $\mathcal{M}_{g,n}$ given by all elements which act as the identity on H. We refer to [Jo83b] and [FM09] for details on the Torelli group. The following theorem summarizes some of the key properties of the Torelli group whose proofs can be found in [FM09, Theorem 7.10], [Jo83a], [Me92]. (See also [MM86]), and [Jo85].)

Theorem 7.1.
1. *The group $\mathcal{I}_{g,n}$ is torsion-free,*
2. *the group $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n = 0, 1$,*
3. *the group $\mathcal{I}_{2,0}$ is a free group on infinitely many generators,*
4. *if $g \geq 3$, then the abelianization of $\mathcal{I}_{g,1}$ is isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b$ for some $a, b \in \mathbb{N}$.*

It is an open question though whether the Torelli group $\mathcal{I}_{g,1}$ is finitely presented for $g \geq 3$ (e.g., see [FM09, Section 7.3]).

Now recall that the action of a homology cylinder on $H = H_1(\Sigma_{g,n})$ also gives rise to an epimorphism $\varphi\colon \mathcal{H}_{g,n} \to \operatorname{Aut}^*(H)$. We now define the *Torelli group $\mathcal{IH}_{g,n}$ of homology cylinders over $\Sigma_{g,n}$* to be the kernel of φ. By Proposition 2.4 we can view the Torelli group $\mathcal{I}_{g,n}$ as a subgroup of $\mathcal{IH}_{g,n}$.

It is an immediate consequence of Proposition 3.5 and Theorem 3.10 that the torsion function gives rise to a homomorphism

$$\tau\colon \mathcal{IH}_{g,n} \longrightarrow Q(H)^\times/N.$$

Note that in this case we do not need to divide $Q(H)^\times$ out by $A = A(H)$ since $\varphi(M)$ is the identity for any $M \in \mathcal{IH}_{g,n}$. We can now prove the analogues of Theorems 1.2 and 1.3.

Theorem 7.2.
1. *If $b_1(\Sigma_{g,n}) > 0$, then there exists an epimorphism*
$$\mathcal{IH}_{g,n} \longrightarrow (\mathbb{Z}/2)^\infty$$
which splits. In particular, the abelianization of $\mathcal{IH}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.
2. *If $g > 1$ or $n > 1$, then there exists an epimorphism*
$$\mathcal{IH}_{g,n} \longrightarrow \mathbb{Z}^\infty.$$
Furthermore, the abelianization of $\mathcal{IH}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

Remark 7.3.
1. Note that provided the genus is at least two Theorem 7.2 (2) holds also for closed surfaces and for surfaces with one boundary component. This is in contrast to the situation in Theorem 1.3.
2. Morita ([Mo08, Corollary 5.2]) used 'trace maps' to show that the abelianization of $\mathcal{IH}_{g,1}, g \geq 1$ has infinite rank. Theorem 7.2 (2) can therefore be seen as an extension of Morita's theorem.

Proof. Part (1) follows immediately from the proof of Theorem 1.2 since the examples $M_i, i \in \mathbb{N}$, provided in the proof are easily seen to lie in $\mathcal{IH}_{g,n}$.

We will now need the following claim:

Claim. Given any $M \in \mathcal{H}_{g,n}$ there exists $M' \in \mathcal{IH}_{g,n}$ with $\tau(M') = \tau(M) \in \mathbb{Z}[H]$.

Indeed, let $M = (M, i_+, i_-)$ be a homology cylinder over $\Sigma_{g,n}$. As we saw in Section 3.3 we have $\varphi(M) \in \operatorname{Aut}^*(H)$, and there exists $\psi \in \mathcal{M}_{g,n}$ such that the induced action on $H_1(\Sigma_{g,n})$ is given by $\varphi(M)$. Now the homology cylinder $M' = (M, i_+, i_- \circ \psi^{-1})$ acts as the identity on H, i.e. M' defines an element in $\mathcal{IH}_{g,n}$. On the other hand it is clear that $\tau(M') = \tau(M)$. This concludes the proof of the claim.

We now turn to the proof of Part (2). First suppose that $n > 1$ and $(g, n) \neq (0, 2)$. Part (2) for this case follows from the proof of Theorem 1.3 since the examples $M(a), a \in \mathbb{N}$, in the proof of Theorem 5.6 can be realized as elements in $\mathcal{IH}_{g,n}$ by the above claim. When $(g, n) = (0, 2)$, following the arguments in Section 2.2 one can easily see that $\mathcal{IH}_{0,2}$ is isomorphic to $\mathcal{C}_{\mathbb{Z}}$ and the desired result follows again from Levine's work [Le69a, Le69b].

Finally suppose that $g > 1$. In this case, for $a \in \mathbb{N}$, we consider the homology cylinder $M(a)$ constructed using Figure 1 modified in the following way: in Figure 1 we have two boundary components, which are connected by x^*, and one hole to which a generator y_1^* is associated. Now remove the second boundary component and replace the first boundary component by a new hole, and denote by x^* a closed curve which 'connects' the two holes. It follows from the discussion of Section 4.2 and from the above claim that there exists a homology cylinder $M(a)$ which lies in the Torelli group $\mathcal{IH}_{g,n}$ and such that

$$\tau(M(a)) = p_a := 1 + (y-1)x + y(y-1)x^2 + \cdots + y^{a-1}(y-1)x^a.$$

Recall that the polynomials $p_a \in \mathbb{Z}[H]$ are irreducible. Also it is evident that the p_a are non-symmetric (i.e., $p_a \neq \bar{p}_a$) and $p_a \neq p_b$ for $a \neq b$. (Note that $p_a \sim p_{\overline{a}}$, and therefore Theorem 1.3 cannot be generalized for this case using our previous method.) To detect $\tau(M(a)) \in Q(H)^\times/N$, for each irreducible $\mu \in \mathbb{Z}[H]$ we use the homomorphism $\Theta_\mu \colon Q(H)^\times/N \to \mathbb{Z}$ defined as in Section 5.1 with the following modification: $e_\mu(p) =$ the exponent of μ in p. Then $\Theta_{p_a}(M(b)) = 1$ if $a = b$ and 0 otherwise. Now following the lines of the proof of Theorem 1.3, Part (2) follows. □

Note that Theorem 7.2 (1) also implies that if $b_1(\Sigma_{g,n}) > 0$, then $\mathcal{IH}_{g,n}$ is neither finitely generated nor finitely related, and it is not torsion-free. Also for the structure of the group $Q(H)^\times/N(H)$, redefining Ψ_λ as we did for Θ_μ in the proof of Theorem 7.2, one can easily obtain the following analogue of Theorem 5.3:

Theorem 7.4. *Suppose H is nontrivial, i.e., $\Sigma_{g,n}$ is neither a sphere nor disk. Then*

$$Q(H)^\times/N(H) \cong \begin{cases} (\mathbb{Z}/2)^\infty & \text{if } g, n \leq 1 \text{ or } (g,n) = (0,2), \\ (\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty & \text{otherwise.} \end{cases}$$

8. Questions

We conclude with a short list of questions:

(1) Study the structure of the kernel of
$$\tau \colon \mathcal{H}_{g,n} \longrightarrow Q(H)^\times/AN \cong (\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty.$$
(2) Does the abelianization of the group $\mathcal{H}_{g,n}$ have infinite rank for $g > 0$ and $n = 0, 1$?
(3) Characterize the image of the homomorphisms $\tau \colon \mathcal{H}_{g,n} \to Q(H)^\times/AN$, $\Psi \colon \mathcal{H}_{g,n} \to (\mathbb{Z}/2)^\infty$ and $\Theta \colon \mathcal{H}_{g,n} \to \mathbb{Z}^\infty$.
(4) Can we realize any element in the image of Ψ by using Theorem 4.2?
(5) Does the abelianization of $\mathcal{H}_{g,n}$ contain 4–torsion? Does it contain any other torsion?
(6) Let $g > 0$ and $n \geq 0$. Does there exist a homomorphism $\mathcal{F} \colon \mathcal{H}^{\text{smooth}}_{g,n} \to A$ onto an abelian group of infinite rank such that the restriction of $\mathcal{F}$ to the kernel of the projection map $\mathcal{H}^{\text{smooth}}_{g,n} \to \mathcal{H}^{\text{top}}_{g,n}$ is also surjective?

(7) Does there exist a monoid homomorphism from $\mathcal{C}_{g,n}$ onto a non-abelian monoid which vanishes on $\mathcal{M}_{g,n}$?
(8) Is the group $\mathcal{H}_{g,n}/\langle \mathcal{M}_{g,n} \rangle$ non-abelian?

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